REALISING THE C^*-ALGEBRA OF A HIGHER-RANK GRAPH AS AN EXEL CROSSED PRODUCT

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Abstract. We use the boundary-path space of a finitely-aligned k-graph Λ to construct a compactly-aligned product system X, and we show that the graph algebra $C^*(\Lambda)$ is isomorphic to the Cuntz-Nica-Pimsner algebra $\mathcal{NO}(X)$. In this setting, we introduce the notion of a crossed product by a semigroup of partial endomorphisms and partially-defined transfer operators by defining it to be $\mathcal{NO}(X)$. We then compare this crossed product with other definitions in the literature.

1. Introduction

In [6], Exel proposed a new definition for a crossed product of a unital C^*-algebra A by an endomorphism α. Exel's definition depends not only on α, but also on the choice of *transfer operator*: a positive continuous linear map $L : A \to A$ satisfying $L(\alpha(a)b) = aL(b)$. We call a triple (A, α, L) an *Exel system*. In his motivating example, Exel finds a family of Exel systems whose crossed products model the Cuntz-Krieger algebras [4]. This marked the first time a crossed product by an endomorphism could successfully model Cuntz-Krieger algebras.

There are two obvious extensions of Exel's construction. Firstly, to a theory of crossed products of *non-unital* C^*-algebras capable of modeling the directed-graph generalisation of the Cuntz-Krieger algebras [20]. In [2], the authors successfully built such a theory, and they realised the graph algebras of locally-finite graphs with no sources as Exel crossed products [2, Theorem 5.1]. The crossed product in question was built from the infinite-path space E^∞ and the shift map σ on E^∞. The hypotheses on E ensure that E^∞ is locally compact, and σ is everywhere defined, and this allows an Exel system to be defined. The other extension of Exel's work is to crossed products by *semigroups* of endomorphisms and transfer operators. In [17], Larsen has a crossed-product construction for dynamical systems (A, P, α, L) in which P is an abelian semigroup, α is an action of P by endomorphisms, and L is an action of P by transfer operators. Exel has also worked in this area with his theory of interaction groups [7, 8].

Motivated by these ideas, we construct a semigroup crossed product that can model the C^*-algebras of the higher-rank graphs, or k-graphs, of Kumjian and Pask [16]. The only restriction we place on the k-graphs Λ whose C^*-algebras we model is a necessary finitely-aligned hypothesis, so our result applies in the fullest possible generality. This does come at a price, however, as without a locally-finite hypothesis, or a restriction on sources, the space of infinite paths is not locally compact. To get a locally-compact

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space we need to consider the bigger *boundary-path space* $\partial\Lambda$, and on this space the shift maps σ_n, $n \in \mathbb{N}^k$, will not in general be everywhere defined. This means we can not form Exel systems, or even a dynamical system in the sense of Larsen [17]. We overcome this problem by first ignoring the crossed-product construction, and focusing on building a *product system*.

A product system of Hilbert A-bimodules over a semigroup P is a semigroup $X = \bigsqcup_{p\in P} X_p$ such that each X_p is a Hilbert A-bimodule, and $x \otimes_A y \mapsto xy$ determines an isomorphism of $X_p \otimes_A X_q$ onto X_{pq} for each $p, q \in P$. Fowler introduced such product systems in [11]. Fowler also defined a Cuntz-Pimsner covariance condition for representations of product systems, and introduced the universal C^*-algebra $\mathcal{O}(X)$ for Cuntz-Pimsner covariant representations of X. This generalised Pimsner's C^*-algebra for a single Hilbert bimodule [19]. In [23], Sims and Yeend looked at the problem of associating a C^*-algebra to product systems which satisfies a gauge-invariant uniqueness theorem, and noted in particular that Fowler's $\mathcal{O}(X)$ will not in general do the job. For a large class of semigroups, and a class of product systems called compactly aligned, Sims and Yeend introduced a covariance condition for representations — called Cuntz-Nica-Pimsner covariance — and a C^*-algebra $\mathcal{NO}(X)$ universal for such representations. A gauge-invariant uniqueness theorem for $\mathcal{NO}(X)$ is proved in [3].

We build from $\partial\Lambda$ and the σ_n topological graphs in the sense of Katsura [14], and then we apply the construction from [14] to get Hilbert $C_0(\partial\Lambda)$-bimodules X_n. We glue the bimodules together to form the *boundary-path product system* X over $\mathbb{N}^k$. This gives a new class of product systems for which the Cuntz-Nica-Pimsner algebra $\mathcal{NO}(X)$ is tractable. The main result in this paper says that for Λ a finitely-aligned k-graph, the graph algebra $C^*(\Lambda)$ is isomorphic to $\mathcal{NO}(X)$. A result, we feel, that gives extra credence to Sims and Yeend's construction, at least in the case for the semigroup $\mathbb{N}^k$. We then construct for each $n \in \mathbb{N}^k$ a partial endomorphism α_n on $C_0(\partial\Lambda)$ and a partially-defined transfer operator L_n, and we *define* the crossed product $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ to be $\mathcal{NO}(X)$. This gives us our desired result: $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k \cong C^*(\Lambda)$.

We begin with some preliminaries in Section 2. We state some necessary definitions from the k-graph literature, and we state the definition of the Cuntz-Krieger algebra of a k-graph. We then state the definitions from [23] needed to make sense of the notion of Cuntz-Nica-Pimsner covariance, and the Cuntz-Nica-Pimsner algebra of a compactly-aligned product system. In Section 3 we construct from a finitely-aligned k-graph Λ the boundary-path product system X. The proof that X is compactly aligned requires substantial detail, so we leave this result for the appendix. In Section 4 we prove the existence of a canonical isomorphism $C^*(\Lambda) \to \mathcal{NO}(X)$. In Section 5 we introduce the crossed product $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$, and we discuss the relationship between this crossed product and the crossed product in [2]; Exel and Royer's crossed product by a partial endomorphism [10]; and Larsen's semigroup crossed product [17].

2. Preliminaries

2.1. **k-graphs and their Cuntz-Krieger algebras.** A higher-rank graph, or k-graph, is a pair (Λ, d) consisting of a countable category Λ and a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the unique factorisation property: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) =$

$m+n$, there are unique elements $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. We now recall some definitions from the k-graph literature; for more details see [5].

For $\lambda, \mu \in \Lambda$ we denote

$$\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) \in \Lambda \times \Lambda : \lambda\alpha = \mu\beta \text{ and } d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}.$$

A k-graph Λ is *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is at most finite for all $\lambda, \mu \in \Lambda$. For each $v \in \Lambda^0$ we denote by $v\Lambda := \{\lambda \in \Lambda : r(\lambda) = v\}$. A subset $E \subseteq v\Lambda$ is *exhaustive* if for every $\mu \in v\Lambda$ there exists a $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. We denote the set of all *finite* exhaustive subsets of Λ by $\mathcal{FE}(\Lambda)$. We denote by $v\mathcal{FE}(\Lambda)$ the set $\{E \in \mathcal{FE}(\Lambda) : E \subseteq v\Lambda\}$.

For each $m \in (\mathbb{N} \cup \{\infty\})^k$ we get a k-graph $\Omega_{k,m}$ through the following construction. The set $\Omega_{k,m}^0 := \{p \in \mathbb{N}^k : p \leq m\}$, and

$$\Omega_k^* := \{(p, q) \in \Omega_{k,m}^0 \times \Omega_{k,m}^0 : p \leq q\}.$$

The range map is given by $r(p, q) = p$; the source map by $s(p, q) = q$; and the degree functor by $d(p, q) = q - p$. Composition is given by $(p, q)(q, r) = (p, r)$.

For k-graph Λ we define a *graph morphism* x to be a degree-preserving functor from $\Omega_{k,m}$ to Λ. The range and degree maps are extended to all graph morphisms $x : \Omega_{k,m} \to \Lambda$ by setting $r(x) := x(0)$ and $d(x) := m$. We define the *boundary-path space* $\partial\Lambda$ to be the set of all graph morphisms x such that for all $n \in \mathbb{N}^k$ with $n \leq d(x)$, and for all $E \in x(n)\mathcal{FE}(\Lambda)$, there exists $\lambda \in E$ such that $x(n, n + d(\lambda)) = \lambda$. We know from [5, Lemmas 5.13] that if $\lambda \in \Lambda x(0)$, then $\lambda x \in \partial\Lambda$. We know from [5, Lemma 5.15] that for each $v \in \Lambda^0$ there exists $x \in v\partial\Lambda = \{x \in \partial\Lambda : r(x) = v\}$.

We recall from [23] the following definition.

Definition 2.1. Let Λ be a finitely-aligned k-graph. A *Cuntz-Krieger family* in a C^*-algebra B is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in B satisfying

(CK1) $\{t_v : v \in \Lambda^0\}$ consists of mutually orthogonal projections;
(CK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(CK3) $t_\lambda^* t_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$; and
(CK4) $\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*) = 0$ for every $v \in \Lambda^0$ and $E \in v\mathcal{FE}(\Lambda)$.

The *Cuntz-Krieger algebra*, or *graph algebra*, $C^*(\Lambda)$ is the universal C^*-algebra generated by a Cuntz-Krieger Λ-family.

2.2. **Product systems and their Cuntz-Nica-Pimsner algebras.** In this subsection we state some key definitions from [23, Sections 2 and 3]; see [23] for more details.

Suppose A is a C^*-algebra, and (G, P) is a quasi-lattice ordered group in the sense that: G is a discrete group and P is a subsemigroup of G; $P \cap P^{-1} = \{e\}$; and with respect to the partial order $p \leq q \iff p^{-1}q \in P$, any two elements $p, q \in G$ which have a common upper bound in P have a least upper bound $p \vee q \in P$. Suppose $X := \bigcup_{p\in P} X_p$ is a product system of Hilbert A-bimodules. For each $p \in P$ and each $x, y \in X_p$ the operator $\Theta_{x,y} : X_p \to X_p$ defined by $\Theta_{x,y}(z) := x \cdot \langle y, z\rangle_A$ is adjointable with $\Theta_{x,y}^* = \Theta_{y,x}$. The span $\mathcal{K}(X_p) := \overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in X_p\}$ is a closed two-sided ideal in $\mathcal{L}(X_p)$ called the *algebra of compact operators on* X_p. For $p, q \in P$ with $e < p \leq q$ there is a homomorphism $\iota_p^q : \mathcal{L}(X_p) \to \mathcal{L}(X_q)$ characterised by

$$\iota_p^q(S)(xy) = (Sx)y \quad \text{for all } x \in X_p, y \in X_{p^{-1}q}. \tag{1}$$

For $p \not\leq q$ we define $\iota_p^q(S) = 0_{\mathcal{L}(X_q)}$ for all $S \in \mathcal{L}(X_p)$. The product system X is called *compactly aligned* if for all $p, q \in P$ such that $p \vee q < \infty$, and for all $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, we have $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q})$.

A *representation* ψ of X in a C^*-algebra B is a map $X \to B$ such that

(1) each $\psi|_{X_p} := \psi_p : X_p \to B$ is linear, and $\psi_e : A \to B$ is a homomorphism;
(2) $\psi_p(x)\psi_y(q) = \psi_{pq}(xy)$ for all $p, q \in P$, $x \in X_p$, and $y \in X_q$; and
(3) $\psi_e(\langle x, y\rangle_A^p) = \psi_p(x)^*\psi_p(y)$ for all $p \in P$, and $x, y \in X_p$.

It follows from Pimsner's results [19] that for each $p \in P$ there is a homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \to B$ satisfying $\psi^{(p)}(\Theta_{x,y}) = \psi_p(x)\psi_p(y)^*$ for all $x, y \in X_p$. A representation ψ of X is *Nica-covariant* if for all $p, q \in P$ and all $S \in \mathcal{K}(X_p), T \in \mathcal{K}(X_q)$ we have

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}\big(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T)\big) & \text{if } p \vee q < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by ϕ_p the homomorphism $A \to \mathcal{L}(X_p)$ implementing the left action of A on X_p. We define $I_e = A$, and for each $q \in P \setminus \{e\}$ we write $I_q := \cap_{e<p\leq q} \ker \phi_p$. We then denote by $\widetilde{X}_q$ the Hilbert A-bimodule

$$\widetilde{X}_q := \bigoplus_{p\leq q} X_p \cdot I_{p^{-1}q},$$

and we denote by $\widetilde{\phi}_q$ the homomorphism implementing the left action of A on $\widetilde{X}_q$. The product system X is said to be $\widetilde{\phi}$*-injective* if every $\widetilde{\phi}_q$ is injective.

For $p, q \in P$ with $p \neq e$ there is a homomorphism $\widetilde{\iota}_p^q : \mathcal{L}(X_p) \to \mathcal{L}(\widetilde{X}_q)$ determined by $S \mapsto \bigoplus_{r\leq q} \iota_p^r(S)$ for all $S \in \mathcal{L}(X_p)$; and characterised by

$$(\widetilde{\iota}_p^q(S)x)(r) = \iota_p^r(S)x(r) \quad \text{for all } x \in \widetilde{X}_q. \tag{2}$$

A representation ψ of a $\widetilde{\phi}$-injective product system X in a C^*-algebra B is *Cuntz-Pimsner covariant* if $\sum_{p\in F} \psi^{(p)}(T_p) = 0_B$ whenever $F \subset P$ is finite, $T_p \in \mathcal{K}(X_p)$ for each $p \in F$, and $\sum_{p\in F} \widetilde{\iota}_p^s(T_p) = 0$ for large s (see [23, Definition 3.8] for the meaning of "for large s"). A representation ψ of a $\tilde{\phi}$-injective product system X is *Cuntz-Nica-Pimsner covariant* if it is both Nica covariant and Cuntz-Pimsner covariant. It is proved in [23, Proposition 3.12] that there exists a C^*-algebra $\mathcal{NO}(X)$, called the *Cuntz-Nica-Pimsner algebra of* X, which is universal for Cuntz-Nica-Pimsner covariant representations of X. We denote the universal Cuntz-Nica-Pimsner representation by $j_X : X \to \mathcal{NO}(X)$.

3. The boundary-path product system of a k-graph

Let Λ be a finitely-aligned k-graph. For $\lambda \in \Lambda$ we denote the set $D_\lambda := \{x \in \partial\Lambda : x(0, d(\lambda)) = \lambda\}$. For $n \in \mathbb{N}^k$ we denote

$$\mathcal{A}^n := \{(\lambda, F) : \lambda \in \Lambda \text{ with } d(\lambda) \geq n,\ F \subseteq s(\lambda)\Lambda \text{ a finite set}\},$$

and $\mathcal{A} := \bigcup_{n\in\mathbb{N}^k} \mathcal{A}^n$. For $(\lambda, F) \in \mathcal{A}$ we denote $D_{\lambda F} := \bigcup_{\nu\in F} D_{\lambda\nu}$. It is proved in [5, Section 5] that the family of sets $\{D_\lambda \setminus D_{\lambda F} : (\lambda, F) \in \mathcal{A}\}$ is a basis of compact and open sets for a Hausdorff topology on $\partial\Lambda$, and $\partial\Lambda$ is a locally compact Hausdorff space. For each $n \in \mathbb{N}^k$ we denote $\partial\Lambda^{\geq n} := \{x \in \partial\Lambda : d(x) \geq n\}$ and $\partial\Lambda^{\not\geq n} := \partial\Lambda \setminus \partial\Lambda^{\geq n}$. We now use the subsets $\partial\Lambda^{\geq n}$ to construct topological graphs in the sense of Katsura [14, 15].

Proposition 3.1. *Let $n \in \mathbb{N}^k$ with $\partial\Lambda^{\geq n} \neq \emptyset$. Denote by σ_n the shift on $\partial\Lambda^{\geq n}$ given by $\sigma_n(x)(m) = x(m+n)$, and $\iota : \partial\Lambda^{\geq n} \to \partial\Lambda$ the inclusion mapping. Then $E_n := (\partial\Lambda, \partial\Lambda^{\geq n}, \sigma_n, \iota)$ is a topological graph.*

Proof. We use the definition of convergence given in [5, Remark 5.6]. Let (x_i) be a sequence in $\partial\Lambda^{\ngeq n}$ converging to x. If $x \in \partial\Lambda^{\geq n}$, then there exists $j \in \{1, \ldots, k\}$ and a subsequence (x_{ik}) of (x_i) such that $d(x_{ik})_j < d(x)_j$ for all x_{ik}. This contradicts that (x_{ik}) converges to x, so we must have $x \in \partial\Lambda^{\ngeq n}$, and hence $\partial\Lambda^{\ngeq n}$ is closed in $\partial\Lambda$. Hence $\partial\Lambda^{\geq n}$ is locally compact.

Let $x \in \partial\Lambda^{\geq n}$. Then $D_{x(0,n)}$ is an open neighbourhood of x, with $D_{x(0,n)} \subseteq \partial\Lambda^{\geq n}$. The map $\sigma_n|_{D_{x(0,n)}} : D_{x(0,n)} \to D_{s(x(0,n))}$ is a bijection, and $\sigma_n(D_{x(0,n)}) = D_{s(x(0,n))}$ is open in $\partial\Lambda$. Now suppose $\lambda \in s(x(0,n))\Lambda$ and $F \subseteq s(\lambda)\Lambda$. Then

$$\sigma_n|_{D_{x(0,n)}}(D_{x(0,n)\lambda} \setminus D_{x(0,n)\lambda F}) = D_\lambda \setminus D_{\lambda F}$$

is open in $D_{s(x(0,n))}$, and

$$\left(\sigma_n|_{D_{x(0,n)}}\right)^{-1}(D_\lambda \setminus D_{\lambda F}) = D_{x(0,n)\lambda} \setminus D_{x(0,n)\lambda F}$$

is open in $D_{x(0,n)}$. Hence, $\sigma_n|_{D_{x(0,n)}}$ is continuous and open, and so it is a homeomorphism of $D_{x(0,n)}$ onto $D_{s(x(0,n))}$. Hence σ_n is a local homeomorphism. We know that ι is continuous, so the result follows. $\square$

We now use Katsura's construction [14] to form Hilbert bimodules. For $f, g \in C_c(\partial\Lambda^{\geq n})$ and $a \in C_0(\partial\Lambda)$, we define

$$(f \cdot a)(x) := f(x)a(\sigma_n(x)) \tag{3}$$

and

$$\langle f, g\rangle_n(x) := \sum_{\sigma_n(y)=x} \overline{f(y)}g(y). \tag{4}$$

We complete $C_c(\partial\Lambda^{\geq n})$ under the norm $\|\cdot\|_n$ given by $\langle\cdot,\cdot\rangle_n$ to get a Hilbert $C_0(\partial\Lambda)$-module $X_n = X(E_n)$. The formula

$$(a \cdot f)(x) := a(\iota(x))f(x) = a(x)f(x), \tag{5}$$

defines an action of $C_0(\partial\Lambda)$ by adjointable operators on X_n, which we denote by $\phi_n : C_0(\partial\Lambda) \to \mathcal{L}(X_n)$, and then X_n becomes a Hilbert $C_0(\partial\Lambda)$-bimodule. For $n \in \mathbb{N}^k$ with $\partial\Lambda^{\geq n} = \emptyset$ we set $X_n := \{0\}$. Note that $X_0 = C_0(\partial\Lambda)$.

Proposition 3.2. *Let $m, n \in \mathbb{N}^k$ with $\partial\Lambda^{\geq m}, \partial\Lambda^{\geq n} \neq \emptyset$. Then the map*

$$\pi : C_c(\partial\Lambda^{\geq m}) \times C_c(\partial\Lambda^{\geq n}) \to C_c(\partial\Lambda^{\geq m+n})$$

given by $\pi(f,g)(x) = f(x)g(\sigma_m(x))$ is a surjective map which induces an isomorphism $\pi_{m,n} : X_m \otimes X_n \to X_{m+n}$ satisfying $\pi_{m,n}(f \otimes g) = f(g \circ \sigma_m)$.

To prove this proposition we need some results. To state these results we use the following notation.

Notation 3.3. (a) Recall from [5, Definition 3.10] that given $\lambda \in \Lambda$ and $E \subseteq r(\lambda)\Lambda$ we denote

$$\operatorname{Ext}(\lambda; E) := \bigcup_{\nu \in E} \{\alpha \in \Lambda : (\alpha, \beta) \in \Lambda^{\min}(\lambda, \nu) \text{ for some } \beta \in \Lambda\}.$$

For $\lambda, \mu \in \Lambda$ we denote $F(\lambda, \mu) := \operatorname{Ext}(\lambda; \{\mu\})$. Since Λ is finitely aligned, $F(\lambda, \mu)$ is a finite subset of $s(\lambda)\Lambda$, and so $(\lambda, F(\lambda, \mu)) \in \mathcal{A}$. We have

$$D_{\lambda F(\lambda,\mu)} = D_{\mu F(\mu,\lambda)}. \tag{6}$$

(b) Let $\lambda, \mu \in \Lambda$ and $x \in \partial\Lambda$ with $d(x) \geq d(\lambda) \vee d(\mu)$. Then we denote by x_λ^μ the path

$$x_\lambda^\mu := x(d(\lambda), d(\lambda) \vee d(\mu)).$$

Lemma 3.4. *Let $(\lambda, F), (\mu, G) \in \mathcal{A}$. Then we have*

$$(D_\lambda \setminus D_{\lambda F}) \cap (D_\mu \setminus D_{\mu G}) = \bigsqcup_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} D_{\lambda\alpha} \setminus D_{\lambda\alpha F_\alpha}, \tag{7}$$

where

$$F_\alpha := \left(\bigcup_{\nu \in F} F(\lambda\alpha, \lambda\nu) \right) \cup \left(\bigcup_{\xi \in G} F(\lambda\alpha, \mu\xi) \right).$$

Proof. The factorisation property ensures that the union in (7) is disjoint.

Let $x \in (D_\lambda \setminus D_{\lambda F}) \cap (D_\mu \setminus D_{\mu G})$. Then $d(x) \geq d(\lambda) \vee d(\mu)$; the pair $(x_\lambda^\mu, x_\mu^\lambda) \in \Lambda^{\min}(\lambda, \mu)$; and $x \in D_{\lambda x_\lambda^\mu}$. Using (6) we have

$$x \in D_{\lambda x_\lambda^\mu F(\lambda x_\lambda^\mu, \lambda\nu)} = D_{\lambda\nu F(\lambda\nu, \lambda x_\lambda^\mu)} \implies x \in D_{\lambda F},$$

which contradicts $x \in D_\lambda \setminus D_{\lambda F}$, so we must have $x \notin D_{\lambda x_\lambda^\mu F(\lambda x_\lambda^\mu, \lambda\nu)}$ for all $\nu \in F$. By symmetry, we also have $x \notin D_{\lambda x_\lambda^\mu F(\lambda x_\lambda^\mu, \mu\xi)}$ for all $\xi \in G$. Hence $x \in D_{\lambda x_\lambda^\mu} \setminus D_{\lambda x_\lambda^\mu F_{x_\lambda^\mu}}$.

Now suppose y is an element of the right-hand-side of (7). So there exists $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ with $y \in D_{\lambda\alpha} \setminus D_{\lambda\alpha F_\alpha}$. We have $y \in D_{\lambda\alpha} \subseteq D_\lambda$. Assume $y \in D_{\lambda\nu}$ for some $\nu \in F$. Then $d(y) \geq d(\lambda\alpha) \vee d(\lambda\nu)$; the pair $(y_{\lambda\alpha}^{\lambda\nu}, y_{\lambda\nu}^{\lambda\alpha}) \in \Lambda^{\min}(\lambda\alpha, \lambda\nu)$; and $y \in D_{\lambda\alpha F(\lambda\alpha, \lambda\nu)} \subseteq D_{\lambda\alpha F_\alpha}$. This is a contradiction, and so $y \notin D_{\lambda\nu}$ for all $\nu \in F$. Hence $y \in D_\lambda \setminus D_{\lambda F}$. By symmetry, we also have $y \in D_\mu \setminus D_{\mu G}$. Hence $y \in (D_\lambda \setminus D_{\lambda F}) \cap (D_\mu \setminus D_{\mu G})$. □

Lemma 3.5. *Let $n \in \mathbb{N}^k$ and $(\lambda, F) \in \mathcal{A}$ with $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\geq n}$. Then we have*

$$D_\lambda \setminus D_{\lambda F} = \bigsqcup_{\mu \in s(\lambda)\Lambda^{d(\lambda) \vee n - d(\lambda)}} D_{\lambda\mu} \setminus D_{\lambda\mu \operatorname{Ext}(\mu; F)}, \tag{8}$$

where $(\lambda\mu, \operatorname{Ext}(\mu; F)) \in \mathcal{A}^n$ for each $\mu \in s(\lambda)\Lambda^{d(\lambda) \vee n - d(\lambda)}$.

Proof. The factorisation property ensures that the union in (8) is disjoint.

Suppose $x \in D_\lambda \setminus D_{\lambda F}$, and consider the path $\mu := x(d(\lambda), d(\lambda) \vee n) \in s(\lambda)\Lambda^{d(\lambda) \vee n - d(\lambda)}$. Then $x \in D_{\lambda\mu}$. If $x \in D_{\lambda\mu \operatorname{Ext}(\mu; F)}$, then there exists $\nu \in F$ and $(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)$ with $x \in D_{\lambda\mu\alpha} = D_{\lambda\nu\beta} \subseteq D_{\lambda\nu} \subseteq D_{\lambda F}$. But this is a contradiction, and so we must have $x \in D_{\lambda\mu} \setminus D_{\lambda\mu \operatorname{Ext}(\mu; F)}$.

Now, let $y \in D_{\lambda\mu} \setminus D_{\lambda\mu \operatorname{Ext}(\mu;F)}$ for some $\mu \in s(\lambda)\Lambda^{d(\lambda)\vee n-d(\lambda)}$. Then $y \in D_\lambda$. If $y \in D_{\lambda\nu}$ for some $\nu \in F$, then the pair

$$\Big(y^{\lambda\nu}_{\lambda\mu}\big(d(\lambda), d(\lambda) + d(\mu) \vee d(\nu)\big), y^{\lambda\mu}_{\lambda\nu}\big(d(\lambda), d(\lambda) + d(\mu) \vee d(\nu)\big)\Big) \in \Lambda^{\min}(\mu, \nu),$$

and $y \in D_{\lambda\mu \operatorname{Ext}(\mu;F)}$. This is a contradiction, and so we must have $y \in D_\lambda \setminus D_{\lambda F}$.

Finally, for each $\mu \in s(\lambda)\Lambda^{d(\lambda)\vee n-d(\lambda)}$ the set $\operatorname{Ext}(\mu; F)$ is finite because F is finite and Λ is finitely aligned. We obviously have $d(\lambda\mu) \geq n$, and so $(\lambda\mu, \operatorname{Ext}(\mu; F)) \in \mathcal{A}^n$. □

Proof of Proposition 3.2. To show that π is surjective we let $f \in C_c(\partial\Lambda^{\geq m+n})$. For each $x \in \operatorname{supp} f$ there exists $(\lambda, F) \in \mathcal{A}$ with $x \in D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\geq m+n}$. So there exists a subset $\mathcal{J} \subseteq \mathcal{A}$ such that $\operatorname{supp} f \subseteq \bigcup_{(\lambda,F)\in\mathcal{J}} D_\lambda \setminus D_{\lambda F}$, where $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\geq m+n}$ for each $(\lambda, F) \in \mathcal{J}$. It follows from Lemma 3.5 that each $D_\lambda \setminus D_{\lambda F}$ is a disjoint union of sets of the form $D_\mu \setminus D_{\mu G}$ with $(\mu, G) \in \mathcal{A}^{m+n}$, and so there exists a subset $\mathcal{J}' \subseteq \mathcal{A}^{m+n}$ such that $\operatorname{supp} f \subseteq \bigcup_{(\mu,G)\in\mathcal{J}'} D_\mu \setminus D_{\mu G}$, where $D_\mu \setminus D_{\mu G} \subseteq \partial\Lambda^{\geq m+n}$ for each $(\mu, G) \in \mathcal{J}'$. Since $\operatorname{supp} f$ is compact, there exists a finite number of pairs $(\mu_j, G_j) \in \mathcal{J}'$ with $\operatorname{supp} f \subseteq \bigcup_{j=1}^h D_{\mu_j} \setminus D_{\mu_j G_j}$. Now for each $1 \leq j \leq h$ let $\lambda_j := \mu_j(m, d(\mu_j))$, and consider the function $\mathcal{X}_{\cup_j D_{\lambda_j} \setminus D_{\lambda_j G_j}} \in C_c(\partial\Lambda^{\geq n})$. Consider also $\tilde{f} \in C_c(\partial\Lambda^{\geq m})$ which is equal to f on $\partial\Lambda^{\geq m+n}$ and zero on the complement. Then we have $\pi(\tilde{f}, \mathcal{X}_{\cup_j D_{\lambda_j} \setminus D_{\lambda_j G_j}}) = f$, and so π maps onto $C_c(\partial\Lambda^{\geq m+n})$.

Routine calculations show that π is bilinear, and so it induces a surjective linear map $\pi_{m,n} : C_c(\partial\Lambda^{\geq m}) \odot C_c(\partial\Lambda^{\geq n}) \to C_c(\partial\Lambda^{\geq m+n})$ satisfying $\pi_{m,n}(f \otimes g)(x) = f(x)g(\sigma_m(x))$. It follows immediately from the formulas (3) and (5) that π preserves the left and right actions.

To see that $\pi_{m,n}$ preserves the inner product, we let $f, h \in C_c(\partial\Lambda^{\geq m})$ and $g, l \in C_c(\partial\Lambda^{\geq n})$. Then for $x \in \partial\Lambda^{\geq m+n}$ we have

$$\begin{aligned}
\langle f \otimes g, h \otimes l\rangle(x) &= \langle \langle h, f\rangle_m \cdot g, l\rangle_n(x) = \sum_{\sigma_n(y)=x} \overline{\langle h, f\rangle_m(y)g(y)}l(y) \\
&= \sum_{\sigma_n(y)=x} \left(\sum_{\sigma_m(z)=y} h(z)\overline{f(z)} \right) \overline{g(y)}l(y) \\
&= \sum_{\sigma_{m+n}(z)=x} \overline{g(\sigma_m(z))}l(\sigma_m(z))\overline{f(z)}h(z). \qquad (9)
\end{aligned}$$

Now

$$\begin{aligned}
\langle \pi_{m,n}(f \otimes g), \pi_{m,n}(h \otimes l)\rangle_{m+n}(x) &= \sum_{\sigma_{m+n}(z)=x} \overline{\pi_{m,n}(f \otimes g)(z)}\pi_{m,n}(h \otimes l)(z) \\
&= \sum_{\sigma_{m+n}(z)=x} \overline{f(z)g(\sigma_m(z))}h(z)l(\sigma_m(z)) \\
&= \langle f \otimes g, h \otimes l\rangle(x),
\end{aligned}$$

and so $\pi_{m,n}$ preserves the inner product. Hence it extends to an isomorphism $\pi_{m,n} : X_m \otimes X_n \to X_{m+n}$. □

Remark 3.6. Suppose $\partial\Lambda^{\geq m}, \partial\Lambda^{\geq n} \neq \emptyset$ and $\partial\Lambda^{\geq m+n} = \emptyset$. We claim that $X_m \otimes X_n = \{0\}$. To see this is true, we assume the contrary. Then there exists $f \in C_c(\partial\Lambda^{\geq m})$ and $g \in C_c(\partial\Lambda^{\geq n})$ with $f \otimes g \neq 0$. It follows from Equation (9) that

$$\langle f \otimes g, f \otimes g\rangle(x) = \sum_{\sigma_{m+n}(z)=x} |f(z)|^2 |g(\sigma_m(z))|^2,$$

and this implies

$$\begin{aligned}\langle f \otimes g, f \otimes g\rangle \neq 0 &\Longleftrightarrow \sigma_{m+n}^{-1}(x) \neq \emptyset \text{ for some } x \in \partial\Lambda\\ &\Longleftrightarrow \partial\Lambda^{\geq m+n} \neq \emptyset.\end{aligned}$$

This is a contradiction, and so we must have $X_m \otimes X_n = \{0\} = X_{m+n}$.

Now suppose that $\partial\Lambda^{\geq m} \neq \emptyset$ and $\partial\Lambda^{\geq n} = \emptyset$. Then we have $\partial\Lambda^{\geq m+n} = \emptyset$, and so $X_n = \{0\} = X_{m+n}$. Then $X_m \otimes X_n = X_m \otimes \{0\} = \{0\} = X_{m+n}$. So we can extend Proposition 3.2 to include all $m, n \in \mathbb{N}^k$, and we think of $\pi_{m,n}$ for m, n as in this remark as the trivial map from $\{0\}$ to itself.

Proposition 3.7. *The family $X := \sqcup_{n\in\mathbb{N}^k} X_n$ of Hilbert bimodules over $C_0(\partial\Lambda)$ with multiplication given by*

$$xy := \pi_{m,n}(x \otimes y) \tag{10}$$

is a product system over $\mathbb{N}^k$.

Proof. We just need to check that $ax = a \cdot x$ and $xa = x \cdot a$ for all $x \in X_n$, $n \in \mathbb{N}^k$ and $a \in C_0(\partial\Lambda)$, but this follows from (3), (5) and the definition of multiplication (10). ☐

We prove that X is compactly aligned in the Appendix.

Given the definition (10) of multiplication within X, we now have the following restatement of Proposition 3.2. This corollary plays an important role in subsequent sections.

Corollary 3.8. *Let $n \in \mathbb{N}^k$ and $h \in C_c(\partial\Lambda^{\geq n})$. Then for every $l, m \in \mathbb{N}^k$ with $n = l+m$, there exists $f \in C_c(\partial\Lambda^{\geq l})$ and $g \in C_c(\partial\Lambda^{\geq m})$ with $h = fg$.*

4. The Cuntz-Nica-Pimsner algebra $\mathcal{NO}(X)$

Recall that we denote by $j_X : X \to \mathcal{NO}(X)$ the universal Cuntz-Nica-Pimsner representation of X. For each $m \in \mathbb{N}^k$ we denote by $j_{X,m}$ the restriction of j_X to X_m. For each $\lambda \in \Lambda$ the set D_λ is closed and open, and so the characteristic function $\mathcal{X}_{D_\lambda} \in C_c(\partial\Lambda^{\geq d(\lambda)}) \subset X_{d(\lambda)}$.

Theorem 4.1. *Let Λ be a finitely aligned k-graph and X be the associated product system of Hilbert bimodules given in* Proposition 3.7. *Denote by $\{s_\lambda : \lambda \in \Lambda\}$ the universal Cuntz-Krieger Λ-family in $C^*(\Lambda)$. There exists an isomorphism $\pi : C^*(\Lambda) \to \mathcal{NO}(X)$ such that $\pi(s_\lambda) = j_{X,d(\lambda)}(\mathcal{X}_{D_\lambda})$.*

To prove this result we first show that $S := \{S_\lambda := j_{X,d(\lambda)}(\mathcal{X}_{D_\lambda}) : \lambda \in \Lambda\}$ is a set of partial isometries in $\mathcal{NO}(X)$ satisfying (CK1) and (CK2). We use the Nica covariance of j_X to show that S satisfies (CK3), and the Cuntz-Pimsner covariance of j_X to show that S satisfies (CK4). The universal property of $C^*(\Lambda)$ then gives us a map $\pi : C^*(\Lambda) \to \mathcal{NO}(X)$ with $\pi(s_\lambda) = j_X(\mathcal{X}_{D_\lambda})$ for each $\lambda \in \Lambda$. We show that S

generates $\mathcal{NO}(X)$, and we use the gauge-invariant uniqueness theorem for $C^*(\Lambda)$ [22, Theorem 4.2] to prove that π is injective.

Proposition 4.2. *The set $S = \{S_\lambda : \lambda \in \Lambda\}$ is a family of partial isometries satisfying* (CK1) *and* (CK2).

Proof. Let $\lambda \in \Lambda$. Using (3) and (4) we get $\mathcal{X}_{D_\lambda} \cdot \langle \mathcal{X}_{D_\lambda}, \mathcal{X}_{D_\lambda} \rangle_{d(\lambda)} = \mathcal{X}_{D_\lambda}$, and it follows that $S_\lambda S_\lambda^* S_\lambda = S_\lambda$. It follows from the properties of characteristic functions that $\{S_v = j_{X,0}(\mathcal{X}_{D_v})\}$ is a set of mutually orthogonal projections, thus (CK1) is satisfied. Relation (CK2) follows from the calculation

$$\begin{aligned}
\mathcal{X}_{D_\lambda}\mathcal{X}_{D_\mu}(x) &= \pi_{d(\lambda),d(\mu)}(\mathcal{X}_{D_\lambda} \otimes \mathcal{X}_{D_\mu})(x) \\
&= \mathcal{X}_{D_\lambda}(x)\mathcal{X}_{D_\mu}(\sigma_{d(\lambda)}(x)) \\
&= \begin{cases} 1 & \text{if } x(0,d(\lambda)) = \lambda \text{ and } x(d(\lambda), d(\lambda)+d(\mu)) = \mu, \\ 0 & \text{otherwise} \end{cases} \\
&= \mathcal{X}_{D_{\lambda\mu}}(x).
\end{aligned}$$
□

Proposition 4.3. *The set S satisfies relation* (CK3)*:*

$$S_\lambda^* S_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_\alpha S_\beta^* \quad \textit{for all } \lambda,\mu \in \Lambda.$$

To prove this proposition we need the next result. For $\lambda, \mu \in \Lambda$ with $d(\lambda) = d(\mu)$ we denote by $\Theta_{\lambda,\mu}$ the rank-one operator $\Theta_{\mathcal{X}_{D_\lambda},\mathcal{X}_{D_\mu}} \in \mathcal{K}(X_{d(\lambda)})$.

Lemma 4.4. *Let $\lambda, \mu \in \Lambda$. Then we have*

$$\iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu}) = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} \Theta_{\lambda\alpha,\mu\beta}. \tag{11}$$

Proof. Let $f \in C_c(\partial\Lambda^{\geq d(\mu)})$ and $g \in C_c(\partial\Lambda^{\geq d(\lambda)\vee d(\mu) - d(\mu)})$. We show that the operators in (11) agree on the product $fg \in C_c(\partial\Lambda^{\geq d(\lambda)\vee d(\mu)})$, and then the result will follow from Corollary 3.8 and the fact that $C_c(\partial\Lambda^{\geq d(\lambda)\vee d(\mu)})$ is dense in $X_{d(\lambda)\vee d(\mu)}$.

We know that for each $\mu \in \Lambda$ we have $\Theta_{\mu,\mu}(f) = \mathcal{X}_{D_\mu} \cdot \langle \mathcal{X}_{D_\mu}, f \rangle_{d(\mu)}$. It follows from a routine calculation using (3) and (4) that $\Theta_{\mu,\mu}(f) = \mathcal{X}_{D_\mu} f$, where $\mathcal{X}_{D_\mu} f$ is a product of functions in $C_c(\partial\Lambda^{\geq d(\mu)})$. It now follows from (1) that

$$\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu})(fg) = (\Theta_{\mu,\mu}(f))g = (\mathcal{X}_{D_\mu} f)g. \tag{12}$$

We now use Corollary 3.8 to factor $(\mathcal{X}_{D_\mu} f)g = hl$, where $h \in C_c(\partial\Lambda^{\geq d(\lambda)})$ and $l \in C_c(\partial\Lambda^{\geq d(\lambda)\vee d(\mu) - d(\lambda)})$. For $x \in \partial\Lambda^{\geq d(\lambda)\vee d(\mu)}$ we have

$$\begin{aligned}
\iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu})(fg)(x) &= \iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})(hl)(x) \\
&= (\mathcal{X}_{D_\lambda} h)l(x) \\
&= \begin{cases} hl(x) & \text{if } x \in D_\lambda, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} fg(x) & \text{if } x \in D_\lambda \cap D_\mu, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

We know from Lemma 3.4 that $D_\lambda \cap D_\mu = \sqcup_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} D_{\lambda\alpha}$. So we have

$$\begin{aligned}\iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu})(fg)(x) &= \begin{cases} fg(x) & \text{if } x \in \sqcup_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} D_{\lambda\alpha}, \\ 0 & \text{otherwise.}\end{cases} \\ &= \left(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} \Theta_{\lambda\alpha,\mu\beta}\right)(fg)(x),\end{aligned}$$

and the result follows. □

Proof of Proposition 4.3. It follows from the Nica covariance of j_X that

$$\begin{aligned}S_\lambda S_\lambda^* S_\mu S_\mu^* &= j_X^{(d(\lambda))}(\Theta_{\lambda,\lambda}) j_X^{(d(\mu))}(\Theta_{\mu,\mu}) \\ &= j_X^{(d(\lambda)\vee d(\mu))}\left(\iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu})\right).\end{aligned}$$

It follows from this equation and Lemma 4.4 that

$$\begin{aligned}S_\lambda\left(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_\alpha S_\beta^*\right)S_\mu^* &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_{\lambda\alpha}S_{\mu\beta}{}^* \\ &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} j_X^{(d(\lambda)\vee d(\mu))}(\Theta_{\lambda\alpha,\mu\beta}) \\ &= j_X^{(d(\lambda)\vee d(\mu))}\left(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} \Theta_{\lambda\alpha,\mu\beta}\right) \\ &= S_\lambda S_\lambda^* S_\mu S_\mu^*.\end{aligned}$$

It then follws that

$$\begin{aligned}S_\lambda^* S_\mu = (S_\lambda^* S_\lambda S_\lambda^*)(S_\mu S_\mu^* S_\mu) &= S_\lambda^*(S_\lambda S_\lambda^* S_\mu S_\mu^*)S_\mu \\ &= S_\lambda^* S_\lambda\left(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_\alpha S_\beta^*\right)S_\mu^* S_\mu \\ &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_{s(\lambda)}S_\alpha(S_{s(\mu)}S_\beta)^* \\ &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_\alpha S_\beta^*.\end{aligned}$$ □

Recall from Section 2.2 that I_n is given by $I_n := \bigcap_{0<m\le n} \ker\phi_m$. To prove that S satisfies (CK4), we need to find families which span dense subspaces of the Hilbert bimodules $X_m \cdot I_{n-m}$, for $m,n \in \mathbb{N}^k$ with $m \le n$. To do this, we must first find families which span dense subspaces of the bimodules X_n and the ideals I_n.

Proposition 4.5. *For each $n \in \mathbb{N}^k$ we have $X_n = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda\setminus D_{\lambda F}} : (\lambda,F)\in\mathcal{A}^n\}$.*

Proof. Let $f \in C_c(\partial\Lambda^{\ge n})$. We can use the same argument as in the beginning of the proof of Proposition 3.2 to write $\operatorname{supp} f \subseteq \bigcup_{j=1}^h D_{\mu_j}\setminus D_{\mu_j G_j}$, where $(\mu_j,G_j)\in\mathcal{A}^n$ and

$D_{\mu_j} \setminus D_{\mu_j G_j} \subseteq \partial\Lambda^{\geq n}$ for each $1 \leq j \leq h$. We now take a partition of unity $\rho_1, \ldots, \rho_h$ subordinate to $\{D_{\mu_j} \setminus D_{\mu_j G_j} : 1 \leq j \leq h\}$, and for $f_j := f\rho_j \in C(D_{\mu_j} \setminus D_{\mu_j G_j})$ we have

$$f = \sum_{j=1}^{h} f_j. \tag{13}$$

Now for each $1 \leq j \leq h$ we have $d(\mu_j) \geq n$. So σ_n is injective on $D_{\mu_j} \setminus D_{\mu_j G_j}$, and hence

$$\|f_j\|_n = \sup\{|f_j(x)| : x \in D_{\mu_j} \setminus D_{\mu_j G_j}\} = \|f_j\|_\infty. \tag{14}$$

Now, it follows from Lemma 3.4 that for each $(\lambda, F) \in \mathcal{A}$ the set

$$\operatorname{span}\{\mathcal{X}_{D_\mu \setminus D_{\mu G}} : (\mu, G) \in \mathcal{A} \text{ and } D_\mu \setminus D_{\mu G} \subseteq D_\lambda \setminus D_{\lambda F}\}$$

is a subalgbera of $C(D_\lambda \setminus D_{\lambda F})$. An application of the Stone-Weierstrass Theorem shows that the closure of that span is equal to $C(D_\lambda \setminus D_{\lambda F})$, and hence each f_j can be uniformly approximated by elements in $\operatorname{span}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : d(\lambda) \geq n\}$. It now follows from (14) that f_j can be uniformly approximated by elements in $\operatorname{span}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : d(\lambda) \geq n\}$ with respect to $\|\cdot\|_n$, and then (13) says that f can be approximated by elements in $\operatorname{span}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : d(\lambda) \geq n\}$ with respect to $\|\cdot\|_n$. The result follows because $C_c(\partial\Lambda \setminus \partial_n)$ is dense in X_n with respect to $\|\cdot\|_n$. □

Definition 4.6. Let $i \in \{1, \ldots, k\}$ and e_i denote the standard basis element of $\mathbb{N}^k$. We say that $(\lambda, F) \in \mathcal{A}$ satisfies condition $K(i)$ if

$$\mu \in s(\lambda)\Lambda \text{ with } d(\mu) \geq e_i \Longrightarrow D_\mu \subseteq D_\nu \text{ for some } \nu \in F.$$

Proposition 4.7. *For each $n \in \mathbb{N}^k$ we have*

$$I_n = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : n_i > 0 \Longrightarrow d(\lambda)_i = 0 \textit{ and } (\lambda, F) \textit{ sat. cond. } K(i)\}.$$

To prove this proposition we need the following result.

Lemma 4.8. *Let $i \in \{1, \ldots, k\}$ and $(\lambda, F) \in \mathcal{A}$. Then $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\ngeq e_i}$ if and only if $d(\lambda)_i = 0$ and (λ, F) satisfies condition $K(i)$. Moreover, we have*

$$\ker \phi_{e_i} = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A},\ d(\lambda)_i = 0 \textit{ and } (\lambda, F) \textit{ satisfies condition } K(i)\}. \tag{15}$$

Proof. Suppose $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\ngeq e_i}$. Then we obviously have $d(\lambda)_i = 0$. Suppose that (λ, F) does not satisfy condition $K(i)$. Then there exists $\mu \in s(\lambda)\Lambda$ with $d(\mu) \geq e_i$, and $x \in D_\mu$ with $x \notin D_\nu$ for all $\nu \in F$. Consider the boundary path λx. We have $d(\lambda x)_i > 0$ and $\lambda x \in D_\lambda \setminus D_{\lambda F}$. But $d(\lambda x)_i > 0 \Longrightarrow \lambda x \in \partial\Lambda^{\geq e_i}$, and this is a contradiction, so (λ, F) satisfies condition $K(i)$.

Now suppose that $d(\lambda)_i = 0$ and (λ, F) satisfies condition $K(i)$. Assume that $D_\lambda \setminus D_{\lambda F} \nsubseteq \partial\Lambda^{\ngeq e_i}$, so there exists $x \in D_\lambda \setminus D_{\lambda F}$ with $x \in \partial\Lambda^{\geq e_i}$. This implies that $d(x)_i > 0$. Consider the edge $\mu := x(d(\lambda), d(\lambda) + e_i)$, which we know exists because $d(\lambda)_i = 0$. We have $\mu \in s(\lambda)\Lambda$ and $d(\mu) = e_i$. The boundary path $\sigma_{d(\lambda)}(x)$ satisfies $\sigma_{d(\lambda)}(x) \in D_\mu$ and $\sigma_{d(\lambda)}(x) \notin D_\nu$ for all $\nu \in F$, and so $D_\mu \nsubseteq D_\nu$ for all $\nu \in F$. But this contradicts that (λ, F) satisfies condition $K(i)$, so we must have $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\ngeq e_i}$.

Now, it follows from Lemma 3.4 and an application of the Stone-Weierstrass Theorem for locally compact spaces that for any open subset U of $\partial\Lambda$ we have

$$C_0(U) = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A} \text{ and } D_\lambda \setminus D_{\lambda F} \subseteq U\}.$$

It follows that

$$\begin{aligned}\ker\phi_{e_i} &= \{a \in C_0(\partial\Lambda) : a|_{\partial\Lambda^{\geq e_i}} = 0\}\\ &= \{a \in C_0(\partial\Lambda) : a|_{\overline{\partial\Lambda^{\geq e_i}}} = 0\}\\ &= C_0(\operatorname{int}\partial\Lambda^{\ngeq e_i})\\ &= \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda\setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A} \text{ and } D_\lambda \setminus D_{\lambda F} \subseteq \operatorname{int}\partial\Lambda^{\ngeq e_i}\}\\ &= \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda\setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A} \text{ and } D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\ngeq e_i}\}\\ &= \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda\setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A},\ d(\lambda)_i = 0,\ (\lambda, F) \text{ satisfies condition } K(i)\}.\end{aligned}$$

□

Proof of Proposition 4.7. We have

$$\begin{aligned}\ker\phi_n &= \{a \in C_0(\partial\Lambda) : a|_{\partial\Lambda^{\geq n}} = 0\} = \{a \in C_0(\partial\Lambda) : a|_{\overline{\partial\Lambda^{\geq n}}} = 0\}\\ &= C_0(\operatorname{int}\partial\Lambda^{\ngeq n}).\end{aligned}$$

Since $m \leq n \implies \partial\Lambda^{\ngeq m} \subseteq \partial\Lambda^{\ngeq n}$, it follows that $m \leq n \Rightarrow \ker\phi_m \subseteq \ker\phi_n$. Hence $I_n = \bigcap_{\{i:\, n_i>0\}} \ker\phi_{e_i}$, and the result now follows from Lemma 4.8. □

Notation 4.9. Let $n \in \mathbb{N}^k$. We define

$$\begin{aligned}\mathcal{I}(I_n) := \{(\lambda, F) \in \mathcal{A} : D_\lambda \setminus D_{\lambda F} \neq \emptyset \text{ and}&\\ n_i > 0 \implies d(\lambda)_i = 0 \text{ and } (\lambda, F) \text{ satisfies condition } K(i)\},&\end{aligned}$$

and for $\mu \in \Lambda$ we write $\mu\mathcal{I}(I_n) := \{(\mu\lambda, F) : (\lambda, F) \in \mathcal{I}(I_n) \text{ with } s(\mu) = r(\lambda)\}$. The reason for introducing this notation is that we can now write

$$I_n = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda\setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{I}(I_n)\}.$$

Proposition 4.10. *Let $m, n \in \mathbb{N}^k$ with $m \leq n$. Then we have*

$$X_m \cdot I_{n-m} = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda\setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A}^m \cap \lambda(0, m)\mathcal{I}(I_{n-m})\}. \tag{16}$$

Proof. We have $X_m \cdot I_{n-m} = \overline{\operatorname{span}}\{x \cdot a : x \in X_m,\ a \in I_{n-m}\}$. To prove that the right-hand side of (16) is contained in the left-hand side, we let $m, n \in \mathbb{N}^k$ with $m \leq n$, and suppose $(\lambda, F) \in \mathcal{A}^m \cap \lambda(0, m)\mathcal{I}(I_{n-m})$. Then $(\lambda(m, d(\lambda)), F) \in \mathcal{I}(I_{n-m})$, and for $x \in \partial\Lambda^{\geq m}$ we have

$$\begin{aligned}\mathcal{X}_{D_\lambda\setminus D_{\lambda F}}(x) &= \begin{cases} 1 & \text{if } x(0, d(\lambda)) = \lambda \text{ and } x(0, d(\lambda\nu)) \neq \lambda\nu \\ & \text{for all } \nu \in F, \\ 0 & \text{otherwise} \end{cases}\\ &= \mathcal{X}_{D_\lambda}(x)\mathcal{X}_{D_{\lambda(m,d(\lambda))}\setminus D_{\lambda(m,d(\lambda))F}}(\sigma_m(x))\\ &= (\mathcal{X}_{D_\lambda} \cdot \mathcal{X}_{D_{\lambda(m,d(\lambda))}\setminus D_{\lambda(m,d(\lambda))F}})(x).\end{aligned}$$

So $\mathcal{X}_{D_\lambda\setminus D_{\lambda F}} = \mathcal{X}_{D_\lambda} \cdot \mathcal{X}_{D_{\lambda(m,d(\lambda))}\setminus D_{\lambda(m,d(\lambda))F}} \in X_m \cdot I_{n-m}$, and it follows that

$$\overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda\setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A}^m \cap \lambda(0, m)\mathcal{I}(I_{n-m})\} \subset X_m \cdot I_{n-m}.$$

It follows from Proposition 4.7 and Proposition 4.5 that

$$X_m \cdot I_{n-m} = \overline{\operatorname{span}}\{\mathcal{X}_{D_\rho\setminus D_{\rho F}} \cdot \mathcal{X}_{D_\tau\setminus D_{\tau G}} : (\rho, F) \in \mathcal{A}^m \text{ and } (\tau, G) \in \mathcal{I}(I_{n-m})\}.$$

So to prove that the left-hand side of (16) is contained in the right-hand side, it suffices to show that for $(\rho, F) \in \mathcal{A}^m$ and $(\tau, G) \in \mathcal{I}(I_{n-m})$ the product $\mathcal{X}_{D_\rho \setminus D_{\rho F}} \cdot \mathcal{X}_{D_\tau \setminus D_{\tau G}}$ is an element of the right-hand side. Since σ_m^{-1} is continuous, the intersection

$$(D_\rho \setminus D_{\rho F}) \cap \sigma_m^{-1}(D_\tau \setminus D_{\tau G}) \tag{17}$$

is an open and compact subset of $D_\rho \setminus D_{\rho F}$. Since it is open, we know there exists a subset $\mathcal{J} \subseteq \mathcal{A}^m$ such that $(D_\rho \setminus D_{\rho F}) \cap \sigma_m^{-1}(D_\tau \setminus D_{\tau G}) = \bigcup_{(\eta,H)\in\mathcal{J}} D_\eta \setminus D_{\eta H}$; since it is compact, there is a finite number, say h, of pairs $(\eta_j, H_j) \in \mathcal{J}$ with

$$(D_\rho \setminus D_{\rho F}) \cap \sigma_m^{-1}(D_\tau \setminus D_{\tau G}) = \bigcup_{j=1}^{h} D_{\eta_j} \setminus D_{\eta_j H_j},$$

We know from Lemma 3.4 that the intersection of sets in the above finite union is a finite, disjoint union of sets of the same form. So it follows that there is a finite number, say l, of pairs $(\mu_j, L_j) \in \mathcal{A}^m$ and constants c_j such that

$$\mathcal{X}_{D_\rho \setminus D_{\rho F}} \cdot \mathcal{X}_{D_\tau \setminus D_{\tau G}} = \mathcal{X}_{(D_\rho \setminus D_{\rho F}) \cap \sigma_m^{-1}(D_\tau \setminus D_{\tau G})} = \sum_{j=1}^{l} c_j \mathcal{X}_{D_{\mu_j} \setminus D_{\mu_j L_j}}. \tag{18}$$

To finish the proof, we need to show that each $(\mu_j, L_j) \in \mu_j(0,m)\mathcal{I}(I_{n-m})$. Suppose $n_i > m_i$ and $d(\mu_j)_i > m_i$. Then for $x \in D_{\mu_j} \setminus D_{\mu_j L_j}$ we have $\sigma_m(x) \in D_\tau \setminus D_{\tau G}$ and $\sigma_m(x)_i > 0$. Since $d(\tau)_i = 0$, there exists a path $\alpha := \sigma_m(x)(d(\tau), d(\tau) + e_i)$ satisfying $\alpha \in s(\tau)\Lambda^{e_i}$. Since (τ, G) satisfies condition $K(i)$, we have $D_\alpha \subseteq D_\xi$ for some $\xi \in G$. But this implies that $\sigma_m(x) = \tau\alpha\sigma_m(x)(d(\tau)+e_i, d(x)) \in D_{\tau\xi} \subseteq D_{\tau G}$, which contradicts $\sigma_m(x) \in D_\tau \setminus D_{\tau G}$. So we must have $d(\mu_j)_i = m_i$.

Now suppose $n_i > m_i$ and there exists an edge $\zeta \in s(\mu_j)\Lambda^{e_i}$ with $D_\zeta \nsubseteq D_\nu$ for any $\nu \in L_j$. Let $x \in s(\zeta)\partial\Lambda$. Then $\mu_j\zeta x \in D_{\mu_j} \setminus D_{\mu_j L_j}$, which implies

$$\sigma_m(\mu_j\zeta x) \in D_\tau \setminus D_{\tau G}. \tag{19}$$

Since $d(\tau)_i = 0$, there exists a path $\beta := \sigma_m(\mu_j\zeta x)(d(\tau), d(\tau)+e_i)$ satisfying $\beta \in s(\tau)\Lambda^{e_i}$. Since (τ, G) satisfies condition $K(i)$, we have $D_\beta \subseteq D_\xi$ for some $\xi \in G$. But this implies that $\sigma_m(\mu_j\zeta x) = \tau\beta\sigma_m(\mu_j\zeta x)(d(\tau) + e_i, d(x)) \in D_{\tau\xi} \subseteq D_{\tau G}$, which contradicts (19). So $D_\zeta \subseteq D_\nu$ for some $\nu \in L_j$, and hence (μ_j, L_j) satisfies condition $K(i)$. □

Notation 4.11. Let $m, n \in \mathbb{N}^k$ with $m \leq n$. We denote

$$\mathcal{I}(X_m \cdot I_{n-m}) := \{(\lambda, F) : D_\lambda \setminus D_{\lambda F} \neq \emptyset,\ (\lambda, F) \in \mathcal{A}^m \cap \lambda(0,m)\mathcal{I}(I_{n-m})\}.$$

So we have

$$X_m \cdot I_{n-m} = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{I}(X_m \cdot I_{n-m})\}.$$

Proposition 4.12. *The set $S = \{S_\lambda : \lambda \in \Lambda\}$ satisfies* (CK4)*:*

$$\prod_{\mu\in\mathcal{F}} (S_v - S_\mu S_\mu^*) = 0$$

for all $v \in \Lambda^0$ and all nonempty finite exhaustive sets $\mathcal{F} \subset r^{-1}(v)$.

To prove this proposition we need the following results. For a finite subset $G \subset \Lambda$ we denote by $\vee d(G)$ the element $\bigvee_{\mu\in G} d(\mu)$ of $\mathbb{N}^k$.

Lemma 4.13. *Let $v \in \Lambda^0$ and $\mathcal{F} \subseteq v\Lambda$ a finite exhaustive set; $n \in \mathbb{N}^k$ with $n \geq \vee d(\mathcal{F})$ and $m \in \mathbb{N}^k$ with $m \leq n$; and $\lambda \in v\Lambda$ and $F \subseteq s(\lambda)\Lambda$ with $(\lambda, F) \in \mathcal{I}(X_m \cdot I_{n-m})$. Then there exists $\eta \in \mathcal{F}$ such that λ extends η.*

Proof. Suppose λ does not extend any element of $\mathcal{F}$. Since $D_\lambda \setminus D_{\lambda F} \neq \emptyset$, there exists a boundary path $x \in D_\lambda \setminus D_{\lambda F}$. Since $\mathcal{F}$ is exhaustive, there exists $\eta \in \mathcal{F}$ with $x(0, d(\eta)) = \eta$. So $x \in D_\eta \cap (D_\lambda \setminus D_{\lambda F})$, and the pair $(x_\lambda^\eta, x_\eta^\lambda) \in \Lambda^{\min}(\lambda, \eta)$. Since λ does not extend η, there exists $i \in \{1, \dots, k\}$ with $d(\lambda)_i < d(\eta)_i$, and hence $d(x_\lambda^\eta) \geq e_i$. Since $m_i \leq d(\lambda)_i < d(\eta)_i \leq n_i$, we know (λ, F) satisfies condition $K(i)$, and hence $D_{x_\lambda^\eta} \subseteq D_\nu$ for some $\nu \in F$. But this implies that $x \in D_{\lambda x_\lambda^\eta} \subseteq D_{\lambda\nu}$, which contradicts the fact $x \notin D_{\lambda F}$. So λ must extend an element of $\mathcal{F}$. $\square$

Lemma 4.14. *Suppose $n \in \mathbb{N}^k$ and $\mu \in \Lambda$ with $d(\mu) \leq n$. Consider the element $\widetilde{x}$ given by $\widetilde{x} := (0, \dots, 0, \mathcal{X}_{D_\lambda \setminus D_{\lambda F}}, 0, \dots, 0) \in \widetilde{X}_n$, where $(\lambda, F) \in \mathcal{I}(X_m \cdot I_{n-m})$ for $m \leq n$. Then we have*

$$\widetilde{\iota}^n_{d(\mu)}(\Theta_{\mu,\mu})(\widetilde{x}) = \begin{cases} \widetilde{x} & \textit{if } \lambda \textit{ extends } \mu, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. It follows from (2) that for $r \leq n$ we have

$$\widetilde{\iota}^n_{d(\mu)}(\Theta_{\mu,\mu})(\widetilde{x})(r) = \iota^r_{d(\mu)}(\Theta_{\mu,\mu})(\widetilde{x}(r)) = \begin{cases} \iota^m_{d(\mu)}(\Theta_{\mu,\mu})(\mathcal{X}_{D_\lambda \setminus D_{\lambda F}}) & \text{if } r = m, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Now assume $m \geq d(\mu)$. A straightforward calculation shows that

$$\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} = \mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))} \setminus D_{\lambda(0,d(\mu))}} \mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))F}}. \tag{21}$$

We also have

$$\begin{aligned} \Theta_{\mu,\mu}(\mathcal{X}_{D_{\lambda(0,d(\mu))}})(x) &= \left(\mathcal{X}_{D_\mu} \cdot \langle \mathcal{X}^*_{D_\mu}, \mathcal{X}_{D_{\lambda(0,d(\mu))}} \rangle_{d(\mu)}\right)(x) \\ &= \mathcal{X}_{D_\mu}(x) \langle \mathcal{X}^*_{D_\mu}, \mathcal{X}_{D_{\lambda(0,d(\mu))}} \rangle_{d(\mu)} (\sigma_{d(\mu)}(x)) \\ &= \begin{cases} \sum_{\sigma_{d(\mu)}(y) = \sigma_{d(\mu)}(x)} \mathcal{X}_{D_\mu}(y) \mathcal{X}_{D_{\lambda(0,d(\mu))}}(y) & \text{if } x(0, d(\mu)) = \mu, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \lambda(0, d(\mu)) = \mu \text{ and } x(0, d(\mu)) = \mu, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathcal{X}_{D_\mu}(x) & \text{if } \lambda \text{ extends } \mu, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathcal{X}_{D_{\lambda(0,d(\mu))}}(x) & \text{if } \lambda \text{ extends } \mu, \\ 0 & \text{otherwise.} \end{cases} \qquad (22) \end{aligned}$$

It now follows from Equations (21) and (22) that

$$
\begin{aligned}
\iota^m_{d(\mu)}(\Theta_{\mu,\mu})(\mathcal{X}_{D_\lambda\setminus D_{\lambda F}}) &= \iota^m_{d(\mu)}(\Theta_{\mu,\mu})(\mathcal{X}_{D_{\lambda(0,d(\mu))}}\mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))}\setminus D_{\lambda(d(\mu),d(\lambda))F}})\\
&= \Theta_{\mu,\mu}(\mathcal{X}_{D_{\lambda(0,d(\mu))}})\mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))}\setminus D_{\lambda(d(\mu),d(\lambda))F}}\\
&= \begin{cases} \mathcal{X}_{D_{\lambda(0,d(\mu))}}\mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))}\setminus D_{\lambda(d(\mu),d(\lambda))F}} & \text{if } \lambda \text{ extends } \mu,\\ 0 & \text{otherwise}\end{cases}\\
&= \begin{cases} \mathcal{X}_{D_\lambda\setminus D_{\lambda F}} & \text{if } \lambda \text{ extends } \mu,\\ 0 & \text{otherwise.}\end{cases} \qquad (23)
\end{aligned}
$$

Equations (20) and (23) now give the result. □

We are now ready to prove that S satisfies relation (CK4). The proof runs through the main argument from the proof of [23, Proposition 5.4].

Proof of Proposition 4.12. Fix $v \in \Lambda^0$ and a finite exhaustive set $\mathcal{F} \subset v\Lambda$. We must show that

$$\prod_{\mu\in\mathcal{F}}(S_v - S_\mu S_\mu^*) = 0.$$

Recall from [21] that for a nonempty subset G of $\mathcal{F}$, $\Lambda^{\min}(G)$ denotes the set $\{\lambda \in \Lambda : d(\lambda) = \vee d(G),\ \lambda \text{ extends } \mu \text{ for all } \mu \in G\}$. Recall also that $\vee\mathcal{F} := \bigcup_{G\subset\mathcal{F}}\Lambda^{\min}(G)$ is finite and is closed under minimal common extensions. We have

$$
\begin{aligned}
\prod_{\mu\in\mathcal{F}}(S_v - S_\mu S_\mu^*) &= S_v + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}}(-1)^{|G|}S_\lambda S_\lambda^*\\
&= j_X^{(0)}(\Theta_{v,v}) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}}(-1)^{|G|}j_X^{(\vee d(G))}(\Theta_{\lambda,\lambda}),
\end{aligned}
$$

where the first equation can be obtained through repeated application of (CK3). Since j_X is Cuntz-Pimsner covariant, it suffices to show that for each $q \in \mathbb{N}^k$ there exists $r \geq q$ such that for all $s \geq r$, we have

$$\widetilde{\iota}^s_0(\Theta_{v,v}) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}}(-1)^{|G|}\widetilde{\iota}^s_{\vee d(G)}(\Theta_{\lambda,\lambda}) = 0.$$

For this, fix $q \in \mathbb{N}^k$, let $r = q \vee (\vee d(\mathcal{F}))$ and fix $s \geq r$. It suffices to show that

$$\Big(\widetilde{\iota}^s_0(\Theta_{v,v}) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}}(-1)^{|G|}\widetilde{\iota}^s_{\vee d(G)}(\Theta_{\lambda,\lambda})\Big)(\widetilde{x}) = 0, \qquad (24)$$

where $\widetilde{x} \in \widetilde{X}_s$ is given by $\widetilde{x} := (0,\ldots,0,\mathcal{X}_{D_\rho\setminus D_{\rho F}},0,\ldots,0)$, for $(\rho, F) \in \mathcal{I}(X_t\cdot I_{s-t})$, $t \leq s$. For any $\mu \in \mathcal{F}$ we have $s \geq d(\mu)$. It then follows from Lemma 4.14 that

$$\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu})(\widetilde{x}) = \begin{cases}\widetilde{x} & \text{if } \rho \text{ extends } \mu,\\ 0 & \text{otherwise.}\end{cases} \qquad (25)$$

Fix a nonempty subset G of $\mathcal{F}$. Then

$$\Big(\prod_{\mu\in G}\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu})\Big)(\widetilde{x})=\begin{cases}\widetilde{x} & \text{if } \rho \text{ extends each } \mu\in G,\\ 0 & \text{otherwise.}\end{cases}$$

The factorisation property implies that ρ extends each $\mu\in G$ if and only if there exists $\lambda\in\Lambda^{\min}(G)$ such that ρ extends λ. The factorisation property also implies that if there does exist such a $\lambda\in\Lambda^{\min}(G)$, then it is necessarily unique. We therefore have

$$\Big(\prod_{\mu\in G}\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu})\Big)(\widetilde{x})=\Big(\sum_{\lambda\in\Lambda^{\min}(G)}\widetilde{\iota}^s_{\vee d(G)}(\Theta_{\mu,\mu})\Big)(\widetilde{x}).$$

Since G was an arbitrary subset of $\mathcal{F}$, we have

$$\begin{aligned}&\Big(\prod_{\mu\in\mathcal{F}}(\widetilde{\iota}^s_0(\Theta_{v,v})-\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu}))\Big)(\widetilde{x})\\
&\qquad=\Big(\widetilde{\iota}^s_0(\Theta_{v,v})+\sum_{\emptyset\neq G\subset\mathcal{F}}\Big((-1)^{|G|}\prod_{\mu\in G}\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu})\Big)\Big)(\widetilde{x})\\
&\qquad=\Big(\widetilde{\iota}^s_0(\Theta_{v,v})+\sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}}(-1)^{|G|}\widetilde{\iota}^s_{\vee d(G)}(\Theta_{\lambda,\lambda})\Big)(\widetilde{x}).\end{aligned}$$

Now we can apply Lemma 4.13 to see that there exists $\eta\in\mathcal{F}$ such that ρ extends η. It now follows from Equation (25) that

$$\begin{aligned}&\Big(\prod_{\mu\in\mathcal{F}}(\widetilde{\iota}^s_0(\Theta_{v,v})-\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu}))\Big)(\widetilde{x})\\
&\quad=\Big(\prod_{\mu\in\mathcal{F}\setminus\{\eta\}}(\widetilde{\iota}^s_0(\Theta_{v,v})-\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu}))\Big)((\widetilde{\iota}^s_0(\Theta_{v,v})-\widetilde{\iota}^s_{d(\eta)}(\Theta_{\eta,\eta})))(\widetilde{x})\\
&\quad=0,\end{aligned}$$

and hence Equation (24) is established. □

Proof of Theorem 4.1. Lemma 4.2, Proposition 4.3 and Proposition 4.12 show that the set $S:=\{S_\lambda=j_X(\mathcal{X}_{D_\lambda}):\lambda\in\Lambda\}$ is a family of partial isometries satisfying the Cuntz-Krieger relations (CK1)–(CK4). It follows from the universal property of $C^*(\Lambda)$ that there exists a homomorphism $\pi:C^*(\Lambda)\to\mathcal{NO}(X)$ such that $\pi(s_\lambda)=j_X(\mathcal{X}_{D_\lambda})$ for each $\lambda\in\Lambda$. We know from [23, Proposition 3.12] that $\mathcal{NO}(X)=\overline{\operatorname{span}}\{j_X(x)j_X(y)^*:x,y\in X\}$. For each $\lambda\in\Lambda$ and $F\subseteq s(\lambda)\Lambda$ we have $\mathcal{X}_{D_\lambda\setminus D_{\lambda F}}=\mathcal{X}_{D_\lambda}-\sum_{\nu\in F}\mathcal{X}_{D_{\lambda\nu}}$, and so

$$j_X(\mathcal{X}_{D_\lambda\setminus D_{\lambda F}})=j_X(\mathcal{X}_{D_\lambda})-j_X\left(\sum_{\nu\in F}\mathcal{X}_{D_{\lambda\nu}}\right)=S_\lambda-\sum_{\nu\in F}S_{\lambda\nu}.$$

It then follows from Proposition 4.5 that S generates $\mathcal{NO}(X)$, and hence π is surjective. It follows from [5, Lemma 5.13(2) and Lemma 5.15] that each $D_\lambda\neq\emptyset$, and hence each $\mathcal{X}_{D_\lambda}\neq 0$. It then follows from [23, Theorem 4.1] that each $S_\lambda\neq 0$. (Note that the quasi-lattice ordered group $(\mathbb{N}^k,\mathbb{Z}^k)$ satisfies [23, Condition (3.5)], and so [23, Theorem 4.1] can indeed be applied.) Since π intertwines the gauge actions of $\mathbb{T}^k$ on $\mathcal{NO}(X)$ and

$C^*(\Lambda)$, the gauge-invariant uniqueness theorem for $C^*(\Lambda)$ [22, Theorem 4.2] implies that π is an isomorphism. □

5. Connections to semigroup crossed products

We begin this section by building a crossed product from a finitely-aligned k-graph Λ. For each $n \in \mathbb{N}^k$ we define a *partial endomorphism* $\alpha_n : C_0(\partial\Lambda) \to C_0(\partial\Lambda^{\geq n})$ given by $\alpha_n(f) = f \circ \sigma_n$. We claim that for $f \in C_c(\partial\Lambda^{\geq n})$ the function $L_n(f)$ given by

$$L_n(f)(x) = \begin{cases} \sum_{\sigma_n(y)=x} f(y) & \text{if } x \in \sigma_n(\partial\Lambda), \\ 0 & \text{otherwise} \end{cases}$$

is well-defined and is an element of $C_c(\partial\Lambda)$. We can cover $\operatorname{supp} f$ with finitely many sets U_i such that $\sigma_n(U_i)$ is open, $\overline{\sigma_n(U_i)}$ is compact, and $\sigma_n|_{U_i}$ is a homeomorphism. The function f must be zero on all but a finite number of points in $\sigma_n^{-1}(x)$. Then near any $x \in \sigma_n(\partial\Lambda)$, $L_n(f) = \sum_{\{i:x\in\sigma_n(U_i)\}} f \circ (\sigma_n|_{U_i})^{-1}$ is a finite sum of continuous functions with compact support. Since $\sigma_n(x)$ is open, $L_n(f) \in C_c(\partial\Lambda)$, and the claim is proved. Routine calculations show that each L_n satisfies the transfer-operator identity: $L_n(\alpha_n(f)g) = fL_n(g)$ for all $f \in C_0(\partial\Lambda)$, $g \in C_c(\partial\Lambda^{\geq n})$. Adapting Exel's construction of a Hilbert bimodule [6] to accommodate the partial maps, and applying it to $(C_0(\partial\Lambda), \alpha_n, L_n)$, gives the Hilbert $C_0(\partial\Lambda)$-bimodule X_n from Section 3. So we consider the boundary-path product system X, and take the suggested route of [2, Section 9] for defining a crossed product for the system $(C_0(\partial\Lambda), \mathbb{N}^k, \alpha, L)$:

Definition 5.1. Let Λ be a finitely-aligned k-graph, and consider the product system X given in Proposition 3.7. We define the *crossed product* $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ to be the Cuntz-Nica-Pimsner algebra $\mathcal{NO}(X)$.

Corollary 5.2. *Let Λ be a finitely-aligned k-graph. Then $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k \cong C^*(\Lambda)$.*

For the remainder of this section we discuss the relationship between the crossed product $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ and the other crossed products in the literature which are given via transfer operators; namely, the non-unital version of Exel's crossed product [2], Exel and Royer's crossed product by a partial endomorphism [10], and Larsen's crossed product for semigroups [17]. The upshot of this discussion is that, when these crossed products can be defined, they coincide with $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$. To be make things clear, we use the following notation.

Notation 5.3. (1) For $(A, \beta, \mathcal{L})$ a dynamical system in the sense of Exel and Royer [10] we denote by $A \rtimes^{\mathrm{ER}}_{\beta,\mathcal{L}} \mathbb{N}$ the crossed product given in [10, Definition 1.6].

(2) For $(A, \beta, \mathcal{L})$ a dynamical system in the sense of [2, 6] we denote by $A \rtimes^{\mathrm{BRV}}_{\beta,\mathcal{L}} \mathbb{N}$ the crossed product given in [2, Section 4].

(3) For P an abelian semigroup and $(A, P, \beta, \mathcal{L})$ a dynamical system in the sense of Larsen [17] we denote by $A \rtimes^{\mathrm{Lar}}_{\beta,\mathcal{L}} P$ the crossed product given in [17, Definition 2.2].

5.1. **Directed graphs.** Suppose Λ is a 1-graph. Then for each $\lambda, \mu \in \Lambda$ we have $|\Lambda^{\min}(\lambda, \mu)| \in \{0, 1\}$, and so Λ is finitely aligned. As shown in [20, Examples 10.1–10.2], Λ is the path category of the directed graph $E := (\Lambda^0, d^{-1}(1), r, s)$. We know from [22, Proposition B.1] that $C^*(\Lambda)$ coincides with the graph algebra $C^*(E)$ as given in [12].

We denote by E^* the set of finite paths in E and by E^∞ the set of infinite paths in E. We define $E^*_{\rm inf} := \{\mu \in E^* : |r^{-1}(s(\mu))| = \infty\}$ and $E^*_{\rm s} := \{\mu \in E^* : r^{-1}(s(\mu)) = \emptyset\}$, so $E^*_{\rm inf}$ is the set of paths whose source is an infinite receiver, and $E^*_{\rm s}$ is the set of paths whose source is a source in E. Then the boundary-path space $\partial\Lambda$ coincides with $\partial E := E^\infty \cup E^*_{\rm inf} \cup E^*_{\rm s}$. We now freely use directed graphs E in place of 1-graphs Λ in Definition 5.1.

Proposition 5.4. *Let E be a directed graph. Then $(C_0(\partial E), \alpha, L)$ is a dynamical system in the sense of* [10]*, and we have $C_0(\partial E) \rtimes_{\alpha,L} \mathbb{N} \cong C_0(\partial E) \rtimes^{\rm ER}_{\alpha,L} \mathbb{N}$.*

To prove this proposition we need the following result.

Proposition 5.5. *Let $(A, \beta, \mathcal{L})$ be a dynamical system in the sense of* [10]*, and consider the Hilbert A-bimodule M constructed in* [10, Section 1]*. Then $A \rtimes^{\rm ER}_{\beta,\mathcal{L}} \mathbb{N}$ is isomorphic to Katsura's Cuntz-Pimsner algebra $\mathcal{O}_M$* [13].

Proof. The arguments in [1, Section 3] (or [2, Section 4]) extend across to this setting, except $A \rtimes^{\rm ER}_{\beta,\mathcal{L}} \mathbb{N}$ is defined by modding out redundancies (a, k) with $a \in (\ker\phi)^\perp \cap \phi^{-1}(\mathcal{K}(M))$ instead of $\overline{A\alpha(A)A} \cap \phi^{-1}(\mathcal{K}(M))$. But $(\ker\phi)^\perp \cap \phi^{-1}(\mathcal{K}(M))$ is precisely the ideal involved in Katsura's definition of $\mathcal{O}_M$ [13, Definition 3.5]. □

Proof of Proposition 5.4. The construction of the Hilbert A-bimodule M from [10] gives X_1. We know from [23, Proposition 5.3] that $\mathcal{NO}(X)$ is isomorphic to Katsura's $\mathcal{O}_{X_1}$. We know from Proposition 5.5 that $C_0(\partial E) \rtimes^{\rm ER}_{\alpha,L} \mathbb{N} \cong \mathcal{O}_M$. So we have

$$C_0(\partial E) \rtimes_{\alpha,L} \mathbb{N} = \mathcal{NO}(X) \cong \mathcal{O}_{X_1} = \mathcal{O}_M \cong C_0(\partial E) \rtimes^{\rm ER}_{\alpha,L} \mathbb{N}.$$ □

5.2. **Locally-finite directed graphs with no sources.** For a locally-finite directed graph $\Lambda := E$ with no sources we have $\partial E = E^\infty$. We denote by σ the backward shift on E^∞, and α_E the endomorphism of $C_0(E^\infty)$ given by $\alpha_E(f) = f \circ \sigma$. So $\alpha_E = \alpha_1$. For each $f \in C_0(E^\infty)$ we denote by $L_E(f)$ the function given by

$$L_E(f)(x) = \begin{cases} \frac{1}{|\sigma^{-1}(x)|} \sum_{\sigma(y)=x} f(y) & \text{if } x \in \sigma(E^\infty), \\ 0 & \text{otherwise} \end{cases}$$

So L_E is the normalised version of L_1. It is proved in [2, Section 2.1] that L_E is a transfer operator for $(C_0(E^\infty), \alpha_E)$.

Proposition 5.6. *Let E be a locally-finite directed graph with no sources. Then we have $C_0(E^\infty) \rtimes_{\alpha,L} \mathbb{N} \cong C_0(E^\infty) \rtimes^{\rm BRV}_{\alpha_E, L_E} \mathbb{N}$.*

Proof. Recall the construction of the Hilbert $C_0(E^\infty)$-bimodule M_{L_E} [2, Section 3], and in particular that $q : C_0(E^\infty) \to M_{L_E}$ denotes the quotient map. Since E is locally finite, the shift σ is proper. We can use this fact to find for each $x \in E^\infty$ an open neighbourhood V of $\sigma(x)$ such that $|\sigma^{-1}(v)| = |\sigma^{-1}(\sigma(x))|$ for each $v \in V$, and it follows that the map $d : E^\infty \to \mathbb{C}$ given by $d(x) = \sqrt{|\sigma^{-1}(\sigma(x))|}$ is continuous. Straightforward calculations show that $U : C_c(E^\infty) \to M_{L_E}$ given by $U(f) = q(df)$ extends to an isomorphism of X_1 onto M_{L_E}. So $\mathcal{O}_{X_1} \cong \mathcal{O}_{M_{L_E}}$. Since E has no sources, the homomorphism $\phi : C_0(E^\infty) \to \mathcal{L}(M_{L_E})$ giving the left action on M_{L_E} is injective, and so $(\ker\phi)^\perp = C_0(E^\infty)$. It then

follows from [2, Corollary 4.2] that $C_0(E^\infty) \rtimes^{\mathrm{BRV}}_{\alpha_E,L_E} \mathbb{N} \cong \mathcal{O}_{M_{L_E}}$. Finally, we know from [23, Proposition 5.3] that $\mathcal{NO}(X) \cong \mathcal{O}_{X_1}$, so we have

$$C_0(E^\infty) \rtimes_{\alpha,L} \mathbb{N} = \mathcal{NO}(X) \cong \mathcal{O}_{X_1} \cong \mathcal{O}_{M_{L_E}} \cong C_0(E^\infty) \rtimes^{\mathrm{BRV}}_{\alpha_E,L_E} \mathbb{N}. \qquad \square$$

5.3. **Regular k-graphs.** We now examine how $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ fits in with the theory of Larsen's semigroup crossed products [17].

If Λ is a row-finite k-graph with no sources, then $\partial\Lambda$ is the set Λ^∞ of all graph morphisms from $\Omega_{k,(\infty,\dots,\infty)}$ to Λ, and the shift maps are everywhere defined. So α is an action by endomorphisms. We say a k-graph Λ is *regular* if it is row-finite with no sources, and there exists $M_1, \dots, M_k \in \mathbb{N} \setminus \{0\}$ such that for each $i \in \{1, \dots, k\}$ we have $|\Lambda^{e_i} v| = M_i$ for all $v \in \Lambda^0$. For each $x \in \Lambda^\infty$ and $n \in \mathbb{N}^k$ define

$$\omega(n, x) := \left|\sigma_n^{-1}(\sigma_n(x))\right|^{-1} = \prod_{i=1}^{k} M_i^{-n_i}.$$

Then for each $f \in C_0(\Lambda^\infty)$ the map $\mathcal{L}_n(f)$ given by

$$\mathcal{L}_n(f)(x) = \begin{cases} \sum_{\sigma_n(y)=x} \omega(n,y) f(y) & \text{if } x \in \sigma_n(\Lambda^\infty), \\ 0 & \text{otherwise} \end{cases}$$

is a transfer operator for $(C_0(\Lambda^\infty), \alpha_n)$. Simple calculations show that

$$\sum_{\sigma_n(y)=x} \omega(n, y) = 1$$

for all $x \in \Lambda^\infty, n \in \mathbb{N}^k$, and that $\omega(m+n, x) = \omega(m, x)\omega(n, \sigma_m(x))$ for all $x \in \Lambda^\infty, m, n \in \mathbb{N}^k$. Hence [9, Proposition 2.2], which still holds in the non-unital setting, gives an action $\mathcal{L}$ of $\mathbb{N}^k$ of transfer operators on $C_0(\Lambda^\infty)$. It follows that $(C_0(\Lambda^\infty), \mathbb{N}^k, \alpha, \mathcal{L})$ is a dynamical system in the sense of Larsen [17, Section 2].

Proposition 5.7. *Let Λ be a regular k-graph. Then we have $C_0(\Lambda^\infty) \rtimes_{\alpha,L} \mathbb{N}^k \cong C_0(\Lambda^\infty) \rtimes^{\mathrm{Lar}}_{\alpha,\mathcal{L}} \mathbb{N}^k$.*

Proof. We apply the construction in [17, Section 3.2] to the dynamical system $(C_0(\Lambda^\infty), \mathbb{N}^k, \alpha, \mathcal{L})$ to form a product system $M = \cup_{n\in\mathbb{N}^k} M_{\mathcal{L}_n}$, and then [17, Proposition 4.3] says $C_0(\Lambda^\infty) \rtimes^{\mathrm{Lar}}_{\alpha,\mathcal{L}} \mathbb{N}^k$ is isomorphic to Fowler's Cuntz-Pimsner algebra $\mathcal{O}(M)$ [11, Proposition 2.9]. Suppose $M_1, \dots, M_k \in \mathbb{N} \setminus \{0\}$ such that for each $i \in \{1, \dots, k\}$ we have $|\Lambda^{e_i} v| = M_i$ for all $v \in \Lambda^0$. For each $n \in \mathbb{N}^k$ denote $M_n := \prod_{i=1}^k M_i^{-n_i}$. Then the map $f \mapsto q_n(\sqrt{M_n} f)$ from $C_c(\Lambda^\infty)$ to $M_{\mathcal{L}_n}$ extends to an isomorphism of X_n onto $M_{\mathcal{L}_n}$. These maps induce an isomorphism of the product systems X and M (observe the formulae for multiplication within X, Proposition 3.2, and M, [17, Equation 3.8]). So $\mathcal{O}(X) \cong \mathcal{O}(M)$.

Recall that each X_n is constructed from the topological graph $(\Lambda^\infty, \Lambda^\infty, \sigma_n, \iota)$, where ι is the inclusion map. It then follows from [14, Proposition 1.24] that each ϕ_n is injective and acts by compact operators. So we can apply [23, Corollary 5.2] to see that $\mathcal{NO}(X)$ coincides with $\mathcal{O}(X)$. So we have

$$C_0(\Lambda^\infty) \rtimes_{\alpha,L} \mathbb{N}^k = \mathcal{NO}(X) = \mathcal{O}(X) \cong \mathcal{O}(M) \cong C_0(\Lambda^\infty) \rtimes^{\mathrm{Lar}}_{\alpha,\mathcal{L}} \mathbb{N}^k. \qquad \square$$

5.4. **Conclusion.** The results in this section justify our decision to define the crossed product $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ to be the Cuntz-Nica-Pimsner algebra $\mathcal{NO}(X)$, and we propose that the same definition is made for a general crossed product by a quasi-lattice ordered semigroup of partial endomorphisms and partially-defined transfer operators. The problem is that Sims and Yeend's Cuntz-Nica-Pimsner algebra is only appropriate for a particular family (containing $\mathbb{N}^k$) of quasi-lattice ordered semigroups. The "correct" definition of a Cuntz-Pimsner algebra of a product system over an arbitrary quasi-lattice ordered semigroup is yet to be found. (See [23, 3] for more discussion.)

6. Appendix

Recall that for (G,P) a quasi-lattice ordered group, and X a product system over P of Hilbert bimodules, we say that X is *compactly aligned* if for all $p,q \in P$ such that $p \vee q < \infty$, and for all $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, we have $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q})$.

Proposition 6.1. *The product system X constructed in* Section 3 *is compactly aligned.*

We start with a definition and some notation.

Definition 6.2. Let $n \in \mathbb{N}^k$. We say that a subset $\mathcal{J} \subseteq \mathcal{A}^n$ is *disjoint* if

$$(\lambda,F),(\mu,G) \in \mathcal{J} \text{ with } (\lambda,F) \neq (\mu,G) \Longrightarrow (D_\lambda \setminus D_{\lambda F}) \cap (D_\mu \setminus D_{\mu G}) = \emptyset.$$

For $(\lambda,F),(\mu,G) \in \mathcal{A}^n$ we write

$$\Theta_{(\lambda,F),(\mu,G)} := \Theta_{\chi_{D_\lambda \setminus D_{\lambda F}},\chi_{D_\mu \setminus D_{\mu G}}} \in \mathcal{K}(X_n).$$

Let $m,n \in \mathbb{N}^k$. To prove Proposition 6.1 we first need to show that for each $(\lambda_1,F_1),(\lambda_2,F_2) \in \mathcal{A}^m$ and $(\mu_1,G_1),(\mu_2,G_2) \in \mathcal{A}^n$ we have

$$\iota_m^{m\vee n}\left(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)}\right)\iota_n^{m\vee n}\left(\Theta_{(\mu_1,G_1),(\mu_2,G_2)}\right) \in \mathcal{K}(X_{m\vee n}).$$

We do this by finding for each $(\alpha,\beta) \in \Lambda^{\min}(\lambda_2,\mu_1)$ finite subsets $\mathcal{H}_{(\alpha,\beta)}, \mathcal{J}_{(\alpha,\beta)} \subseteq \mathcal{A}^{m\vee n}$ such that $\sqcup_{(\alpha,\beta)}\mathcal{H}_{(\alpha,\beta)}$ and $\sqcup_{(\alpha,\beta)}\mathcal{J}_{(\alpha,\beta)}$ are disjoint, and

$$\begin{aligned}\iota_m^{m\vee n}&\left(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)}\right)\iota_n^{m\vee n}\left(\Theta_{(\mu_1,G_1),(\mu_2,G_2)}\right)\\ &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)} \sum_{\substack{(\kappa,H)\in\mathcal{H}_{(\alpha,\beta)}\\(\omega,J)\in\mathcal{J}_{(\alpha,\beta)}}} \Theta_{(\kappa,H),(\omega,J)}.\end{aligned} \tag{26}$$

To find the correct $\mathcal{H}_{(\alpha,\beta)}$ and $\mathcal{J}_{(\alpha,\beta)}$, we evaluate both sides of (26) on products fg, where $f \in C_c(\partial\Lambda^{\geq n})$ and $g \in C_c(\partial\Lambda^{\geq m\vee n-n})$. For the left-hand-side of (26) we use (1) and Corollary 3.8 to factor

$$\iota_n^{m\vee n}\left(\Theta_{(\mu_1,G_1),(\mu_2,G_2)}\right)(fg) = \Theta_{(\mu_1,G_1),(\mu_2,G_2)}(f)g = hl,$$

where $h \in C_c(\partial\Lambda^{\geq m})$ and $l \in C_c(\partial\Lambda^{\geq m\vee n-m})$. Then for $x \in \partial\Lambda^{\geq m\vee n}$ we have

$$\begin{aligned}
&\iota_m^{m\vee n}\left(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)}\right)\iota_n^{m\vee n}\left(\Theta_{(\mu_1,G_1),(\mu_2,G_2)}\right)(fg)(x)\\
&= \iota_m^{m\vee n}\left(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)}\right)(hl)(x)\\
&= \Theta_{(\lambda_1,F_1),(\lambda_2,F_2)}(h)l(x)\\
&= \mathcal{X}_{D_{\lambda_1}\setminus D_{\lambda_1F_1}}(x)\langle \mathcal{X}_{D_{\lambda_2}\setminus D_{\lambda_2F_2}}, h\rangle_m(\sigma_m(x))l(\sigma_m(x))\\
&= \mathcal{X}_{D_{\lambda_1}\setminus D_{\lambda_1F_1}}(x)\left(\sum_{\sigma_m(y)=\sigma_m(x)}\overline{\mathcal{X}_{D_{\lambda_2}\setminus D_{\lambda_2F_2}}(y)}h(y)\right)l(\sigma_m(x))\\
&= \begin{cases} hl(\lambda_2(0,m)\sigma_m(x)) & \text{if } x \in (D_{\lambda_1}\setminus D_{\lambda_1F_1})\cap\sigma_m^{-1}(D_{\lambda_2(m,d(\lambda_2))}\setminus D_{\lambda_2(m,d(\lambda_2))F_2}),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

A similar calculation to the one above gives

$$\begin{aligned}
&hl(\lambda_2(0,m)\sigma_m(x))\\
&= \Theta_{(\mu_1,G_1),(\mu_2,G_2)}(f)g(\lambda_2(0,m)\sigma_m(x))\\
&= \begin{cases} fg(\mu_2(0,n)\sigma_n(\lambda_2(0,m)\sigma_m(x))) & \text{if } \lambda_2(0,m)\sigma_m(x) \in (D_{\mu_1}\setminus D_{\mu_1G_1})\cap\\ & \qquad \sigma_n^{-1}(D_{\mu_2(n,d(\mu_2))}\setminus D_{\mu_2(n,d(\mu_2))G_2}),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

So we label conditions

$$x \in (D_{\lambda_1}\setminus D_{\lambda_1F_1})\cap\sigma_m^{-1}(D_{\lambda_2(m,d(\lambda_2))}\setminus D_{\lambda_2(m,d(\lambda_2))F_2}), \tag{27}$$

and

$$\lambda_2(0,m)\sigma_m(x) \in (D_{\mu_1}\setminus D_{\mu_1G_1})\cap\sigma_n^{-1}(D_{\mu_2(n,d(\mu_2))}\setminus D_{\mu_2(n,d(\mu_2))G_2}), \tag{28}$$

and then we have

$$\begin{aligned}
&\iota_m^{m\vee n}\left(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)}\right)\iota_n^{m\vee n}\left(\Theta_{(\mu_1,G_1),(\mu_2,G_2)}\right)(fg)(x)\\
&\qquad = \begin{cases} fg(\mu_2(0,n)\sigma_n(\lambda_2(0,m)\sigma_m(x))) & \text{if } x \text{ satisfies (27) and(28)},\\ 0 & \text{otherwise.}\end{cases}
\end{aligned} \tag{29}$$

Now, for each $(\alpha,\beta) \in \Lambda^{\min}(\lambda_2,\mu_1)$, $(\kappa,H) \in \mathcal{H}_{(\alpha,\beta)}$ and $(\omega,J) \in \mathcal{J}_{(\alpha,\beta)}$ we have

$$\begin{aligned}
&\Theta_{(\kappa,H),(\omega,J)}(fg)(x)\\
&\qquad = \mathcal{X}_{D_\kappa\setminus D_{\kappa H}}(x)\langle \mathcal{X}_{D_\omega\setminus D_{\omega J}}, fg\rangle_{m\vee n}(\sigma_{m\vee n}(x))\\
&\qquad = \mathcal{X}_{D_\kappa\setminus D_{\kappa H}}(x)\left(\sum_{\sigma_{m\vee n}(y)=\sigma_{m\vee n}(x)}\overline{\mathcal{X}_{D_\omega\setminus D_{\omega J}}(y)}fg(y)\right)\\
&\qquad = \begin{cases} fg(\tau(0,m\vee n)\sigma_{m\vee n}(x)) & \text{if } x \in (D_\kappa\setminus D_{\kappa H})\cap\sigma_{m\vee n}^{-1}(D_\omega\setminus D_{\omega J}),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Since $\sqcup_{(\alpha,\beta)}\mathcal{H}_{(\alpha,\beta)}$ and $\sqcup_{(\alpha,\beta)}\mathcal{J}_{(\alpha,\beta)}$ are disjoint, we have

$$\left(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)}\ \sum_{\substack{(\kappa,H)\in\mathcal{H}\\(\tau,J)\in\mathcal{J}}}\Theta_{(\kappa,H),(\omega,J)}\right)(fg)(x) \tag{30}$$
$$=\begin{cases} fg(\tau(0,m\vee n)\sigma_{m\vee n}(x)) & \text{if } x\in\sqcup_{\substack{(\alpha,\beta)\\(\kappa,H),(\omega,J)}}(D_\kappa\setminus D_{\rho H})\cap\sigma_{m\vee n}^{-1}(D_\omega\setminus D_{\omega J}),\\ 0 & \text{otherwise.}\end{cases}$$

Equation (26) now follows from (29), (30) and the following lemma.

Lemma 6.3. *Let $m,n\in\mathbb{N}^k$, and suppose the pairs $(\lambda_1,F_1),(\lambda_2,F_2)\in\mathcal{A}^m$ and $(\mu_1,G_1),(\mu_2,G_2)\in\mathcal{A}^n$. Then for each pair $(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)$ there exists finite and disjoint subsets $\mathcal{H}_{(\alpha,\beta)},\mathcal{J}_{(\alpha,\beta)}\subseteq\mathcal{A}^{m\vee n}$ such that $x\in\partial\Lambda^{\geq m\vee n}$ satisfies Equations (27) and (28) if and only if*

$$x\in\bigsqcup_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)}\ \bigsqcup_{\substack{(\kappa,H)\in\mathcal{H}_{(\alpha,\beta)}\\(\omega,J)\in\mathcal{J}_{(\alpha,\beta)}}}(D_\kappa\setminus D_{\kappa H})\cap\sigma_{m\vee n}^{-1}(D_\omega\setminus D_{\omega J}). \tag{31}$$

Moreover, if x satisfies (27) and (28) and $x\in(D_\kappa\setminus D_{\kappa H})\cap\sigma_{m\vee n}^{-1}(D_\omega\setminus D_{\omega J})$, then we have

$$\mu_2(0,n)\sigma_n(\lambda_2(0,m)\sigma_m(x))=\omega(0,m\vee n)\sigma_{m\vee n}(x).$$

Proof. Recall that for $\lambda,\mu\in\Lambda$ we denote by

$$F(\lambda,\mu)=\{\alpha\in\Lambda:(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)\text{ for some }\beta\in\Lambda\}.$$

Let $(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)$. For each $(\gamma,\delta)\in\Lambda^{\min}(\lambda_1(m,d(\lambda_1)),\lambda_2(m,d(\lambda_2))\alpha)$ we define

$$H_{\gamma,\alpha}:=\left(\bigcup_{\nu\in F_1}F(\lambda_1\gamma,\lambda_1\nu)\right)\cup\left(\bigcup_{\zeta\in F_2}F(\lambda_2(m,d(\lambda_2))\alpha\delta,\lambda_2(m,d(\lambda_2))\zeta)\right)$$
$$\cup\left(\bigcup_{\eta\in G_1}F(\mu_1\beta\delta,\mu_1\eta)\right),$$

and

$$\mathcal{H}_{(\alpha,\beta)}:=\{(\lambda_1\gamma,H_{\gamma,\alpha})\in\mathcal{A}^{m\vee n}:(\gamma,\delta)\in\Lambda^{\min}(\lambda_1(m,d(\lambda_1)),\lambda_2(m,d(\lambda_2))\alpha)\}.$$

For each $(\rho,\tau)\in\Lambda^{\min}(\mu_2(n,d(\mu_2)),\mu_1(n,d(\mu_1))\beta)$ we define

$$J_{\rho,\beta}:=\left(\bigcup_{\xi\in G_2}F(\mu_2\rho,\mu_2\xi)\right)\cup\left(\bigcup_{\eta\in G_1}F(\mu_1(n,d(\mu_1))\beta\tau,\mu_1(n,d(\mu_1))\eta)\right)$$
$$\cup\left(\bigcup_{\zeta\in F_2}F(\lambda_2\alpha\tau,\lambda_2\zeta)\right),$$

and

$$\mathcal{J}_{(\alpha,\beta)}:=\{(\mu_2\rho,H_{\rho,\beta})\in\mathcal{A}^{m\vee n}:(\rho,\tau)\in\Lambda^{\min}(\mu_2(n,d(\mu_2)),\mu_1(n,d(\mu_1))\beta)\}.$$

The sets $\mathcal{H}_{(\alpha,\beta)}$ and $\mathcal{J}_{(\alpha,\beta)}$ are finite sets because Λ is finitely aligned. Since the paths in the elements of $\mathcal{H}_{(\alpha,\beta)}$ are of the same length, the factorisation property ensures that each $\mathcal{H}_{(\alpha,\beta)}$ is disjoint. For the same reason, each $\mathcal{J}_{(\alpha,\beta)}$ is disjoint. This explains why the second union in (31) is a disjoint union. Moreover, the sets $\sqcup_{(\alpha,\beta)}\mathcal{H}_{(\alpha,\beta)}$ and $\sqcup_{(\alpha,\beta)}\mathcal{J}_{(\alpha,\beta)}$ are disjoint, and hence why the first union in (31) is a disjoint union.

To prove the 'only if' part of the statement, we assume $x \in \partial\Lambda^{\geq m\vee n}$ satisfies (27) and (28). We have to find pairs

$$\begin{aligned}
&(\alpha,\beta) \in \Lambda^{\min}(\lambda_2,\mu_1),\\
&(\gamma,\delta) \in \Lambda^{\min}(\lambda_1(m,d(\lambda_1)),\lambda_2(m,d(\lambda_2))\alpha), \text{ and}\\
&(\rho,\tau) \in \Lambda^{\min}(\mu_2(n,d(\mu_2)),\mu_1(n,d(\mu_1))\beta)
\end{aligned}$$

such that

(a) $x \in D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}}$, and
(b) $\sigma_{m\vee n}(x) \in D_{\mu_2\rho(m\vee n,d(\mu_2\rho))} \setminus D_{\mu_2\rho(m\vee n,d(\mu_2\rho))J_{\rho,\beta}}$.

Now, we know from (27) and (28) that $\lambda_2(0,m)\sigma_m(x) \in D_{\lambda_2}\cap D_{\mu_1}$, so we take

$$(\alpha,\beta) := \big(\lambda_2(0,m)\sigma_m(x)^{\mu_1}_{\lambda_2}, \lambda_2(0,m)\sigma_m(x)^{\lambda_2}_{\mu_1}\big) \in \Lambda^{\min}(\lambda_2,\mu_1). \tag{32}$$

We know from (27) and (28) that $\sigma_m(x) \in D_{\lambda_1(m,d(\lambda_1))}\cap D_{\lambda_2(m,d(\lambda_2))\alpha}$, so we define (γ,δ) to be the pair

$$\Big(\sigma_m(x)^{\lambda_2(m,d(\lambda_2))\alpha}_{\lambda_1(m,d(\lambda_1))}, \sigma_m(x)^{\lambda_1(m,d(\lambda_1))}_{\lambda_2(m,d(\lambda_2))\alpha}\Big) \in \Lambda^{\min}(\lambda_1(m,d(\lambda_1)),\lambda_2(m,d(\lambda_2))\alpha). \tag{33}$$

We now have $\sigma_m(x) \in D_{\lambda_1(m,d(\lambda_1))\gamma}$, and this along with (27) implies that $x \in D_{\lambda_1\gamma}$. We also have

$$x \in D_{\lambda_1\gamma} \text{ and } x \notin D_{\lambda_1 F_1} \Longrightarrow x \notin D_{\lambda_1\gamma\nu'} \text{ for all } \nu' \in \bigcup_{\nu\in F_1} F(\lambda_1\gamma,\lambda_1\nu); \tag{34}$$

$$\begin{aligned}
&\sigma_m(x) \in D_{\lambda_2(m,d(\lambda_2))\alpha\delta} \text{ and } \sigma_m(x) \notin D_{\lambda_2(m,d(\lambda_2))F_2}\\
&\Longrightarrow \sigma_m(x) \notin D_{\lambda_2(m,d(\lambda_2))\alpha\delta\zeta'} \text{ for all } \zeta' \in \bigcup_{\zeta\in F_2} F(\lambda_2(m,d(\lambda_2))\alpha\delta,\lambda_2(m,d(\lambda_2))\zeta)\\
&\Longleftrightarrow \sigma_m(x) \notin D_{\lambda_1(m,d(\lambda_1))\gamma\zeta'} \text{ for all } \zeta' \in \bigcup_{\zeta\in F_2} F(\lambda_2(m,d(\lambda_2))\alpha\delta,\lambda_2(m,d(\lambda_2))\zeta)\\
&\Longleftrightarrow x \notin D_{\lambda_1\gamma\zeta'} \text{ for all } \bigcup_{\zeta\in F_2} F(\lambda_2(m,d(\lambda_2))\alpha\delta,\lambda_2(m,d(\lambda_2))\zeta); \qquad (35)
\end{aligned}$$

and

$$\begin{aligned}\lambda_2(0,m)\sigma_m(x) \in & D_{\lambda_2\alpha\delta} \text{ and } \lambda_2(0,m)\sigma_m(x) \notin D_{\mu_1 G_1} \\ & \Longrightarrow \lambda_2(0,m)\sigma_m(x) \notin D_{\lambda_2\alpha\delta\eta'} \text{ for all } \eta' \in \bigcup_{\eta\in G_1} F(\mu_1\beta\delta, \mu_1\eta) \\ & \Longleftrightarrow \sigma_m(x) \notin D_{\lambda_2(m,d(\lambda_2))\alpha\delta\eta'} \text{ for all } \eta' \in \bigcup_{\eta\in G_1} F(\mu_1\beta\delta, \mu_1\eta) \\ & \Longleftrightarrow \sigma_m(x) \notin D_{\lambda_1(m,d(\lambda_1))\gamma\eta'} \text{ for all } \eta' \in \bigcup_{\eta\in G_1} F(\mu_1\beta\delta, \mu_1\eta) \\ & \Longleftrightarrow x \notin D_{\lambda_1\gamma\eta'} \text{ for all } \eta' \in \bigcup_{\eta\in G_1} F(\mu_1\beta\delta, \mu_1\eta). \qquad (36)\end{aligned}$$

It follows from (34), (35) and (36) that $x \notin D_{\lambda_1\gamma H_{\gamma,\alpha}}$, and so (a) is satisfied.

We have $\sigma_n(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_1(n,d(\mu_1))\beta}$, and it follows from (28) that $\sigma_n(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_2(n,d(\mu_2))}$. So we take

$$\begin{aligned}(\rho,\tau) := & \Big(\sigma_n(\lambda_2(0,m)\sigma_m(x))^{\mu_1(n,d(\mu_1))\beta}_{\mu_2(n,d(\mu_2))}, \sigma_n(\lambda_2(0,m)\sigma_m(x))^{\mu_2(n,d(\mu_2))}_{\mu_1(n,d(\mu_1))\beta}\Big) \\ & \in \Lambda^{\min}(\mu_2(n,d(\mu_2)), \mu_1(n,d(\mu_1))\beta), \qquad (37)\end{aligned}$$

and we have

$$\begin{aligned}\sigma_n(\lambda_2(0,m)\sigma_m(x)) \in & D_{\mu_2(n,d(\mu_2))\rho} \\ & \Longrightarrow \sigma_{m\vee n}(x) = \sigma_{m\vee n}(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_2\rho(m\vee n, d(\mu_2\rho))}.\end{aligned}$$

Suppose for contradiction that there exists $\xi \in G_2$ and a pair (ξ', ξ'') in the set $\Lambda^{\min}(\mu_2\rho, \mu_2\xi)$ with $\sigma_{m\vee n}(x) \in D_{\mu_2\rho(m\vee n, d(\mu_2\rho))\xi'}$. Then it follows from (37) that

$$\begin{aligned}\sigma_n(\lambda_2(0,m)\sigma_m(x)) &= \sigma_n(\lambda_2(0,m)\sigma_m(x))(0, m\vee n - n)\sigma_{m\vee n}(x) \\ &= \mu_2(n,d(\mu_2))\rho(0, m\vee n - n)\sigma_{m\vee n}(x) \\ &= \mu_2\rho(n, m\vee n)\sigma_{m\vee n}(x) \\ &\in D_{\mu_2(n,d(\mu_2))\rho\xi'} \\ &= D_{\mu_2(n,d(\mu_2))\xi\xi''} \\ &\subseteq D_{\mu_2(n,d(\mu_2))G_2}.\end{aligned}$$

This contradicts Equation (28), and so we must have

$$\sigma_{m\vee n}(x) \notin D_{\mu_2\rho(m\vee n, d(\mu_2\rho))\xi'} \text{ for all } \xi' \in \bigcup_{\xi\in G_2} F(\mu_2\rho, \mu_2\xi). \qquad (38)$$

Similar arguments show that

$$\sigma_{m\vee n}(x) \notin D_{\mu_2\rho(m\vee n, d(\mu_2\rho))\eta'}, \text{ for all } \eta' \in \bigcup_{\eta\in G_1} F(\mu_1(n,d(\mu_1))\beta\tau, \mu_1(n,d(\mu_1))\eta), \qquad (39)$$

and

$$\sigma_{m\vee n}(x) \notin D_{\mu_2\rho(m\vee n, d(\mu_2\rho))\eta'}, \text{ for all } \zeta' \in \bigcup_{\zeta\in F_2} F(\lambda_2\alpha\tau, \lambda_2\zeta). \qquad (40)$$

It follows from (38), (39) and (40) that $\sigma_{m\vee n}(x) \notin D_{\mu_2\rho(m\vee n,d(\mu_2\rho))J_{\rho,\beta}}$, and so (b) is satisfied.

To prove the 'if' part of the statement, we assume there exists

$$\begin{aligned}&(\alpha,\beta) \in \Lambda^{\min}(\lambda_2,\mu_1),\\&(\gamma,\delta) \in \Lambda^{\min}(\lambda_1(m,d(\lambda_1)),\lambda_2(m,d(\lambda_2))\alpha), \text{ and}\\&(\rho,\tau) \in \Lambda^{\min}(\mu_2(n,d(\mu_2)),\mu_1(n,d(\mu_1))\beta),\end{aligned}$$

such that

$$x \in \big(D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}}\big) \cap \sigma_{m\vee n}^{-1}(D_{\mu_2\rho(m\vee n,d(\mu_2\rho))} \setminus D_{\mu_2\rho(m\vee n,d(\mu_2\rho))J_{\rho,\beta}}).$$

We have

$$x \in D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}} \Longrightarrow x \in D_{\lambda_1} \setminus D_{\lambda_1 F_1},$$

and

$$\begin{aligned}x \in D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}} &\Longrightarrow \sigma_m(x) \in D_{\lambda_1(m,d(\lambda_1))\gamma} \setminus D_{\lambda_1(m,d(\lambda_1))\gamma H_{\gamma,\alpha}}\\&\Longleftrightarrow \sigma_m(x) \in D_{\lambda_2(m,d(\lambda_2))\alpha\delta} \setminus D_{\lambda_2(m,d(\lambda_2))\alpha\delta H_{\gamma,\alpha}}\\&\Longleftrightarrow \sigma_m(x) \in D_{\lambda_2(m,d(\lambda_2))} \setminus D_{\lambda_2(m,d(\lambda_2))F_2}.\end{aligned}$$

So (27) is satisfied. We have

$$\begin{aligned}x \in D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}} &\Longrightarrow \sigma_m(x) \in D_{\lambda_1(m,d(\lambda_1))\gamma} \setminus D_{\lambda_1(m,d(\lambda_1))\gamma H_{\gamma,\alpha}}\\&\Longleftrightarrow \sigma_m(x) \in D_{\lambda_2(m,d(\lambda_2))\alpha\delta} \setminus D_{\lambda_2(m,d(\lambda_2))\alpha\delta H_{\gamma,\alpha}}\\&\Longrightarrow \lambda_2(0,m)\sigma_m(x) \in D_{\lambda_2\alpha\delta} \setminus D_{\lambda_2\alpha\delta H_{\gamma,\alpha}}\\&\Longleftrightarrow \lambda_2(0,m)\sigma_m(x) \in D_{\mu_1\beta\delta} \setminus D_{\mu_1\beta\delta H_{\gamma,\alpha}}\\&\Longleftrightarrow \lambda_2(0,m)\sigma_m(x) \in D_{\mu_1} \setminus D_{\mu_1 G_1}.\end{aligned}$$

We have

$$\begin{aligned}x \in D_{\lambda_1\gamma} \Longrightarrow \lambda_2(0,m)\sigma_m(x)(n,m\vee n) &= \big(\lambda_2(0,m)\lambda_1(m,d(\lambda_1))\gamma\big)(n,m\vee n)\\&= \big(\lambda_2(0,m)\lambda_2(m,d(\lambda_2))\alpha\delta\big)(n,m\vee n)\\&= \lambda_2\alpha\delta(n,m\vee n)\\&= \lambda_2\alpha(n,m\vee n)\\&= \mu_1\beta(n,m\vee n)\\&= \big(\mu_1(n,d(\mu_1))\beta\big)(n,m\vee n)\\&= \big(\mu_1(n,d(\mu_1))\beta\tau\big)(n,m\vee n)\\&= \big(\mu_2(n,d(\mu_2))\rho\big)(n,m\vee n).\end{aligned}$$

It follows that

$$\begin{aligned}\sigma_n(\lambda_2(0,m)\sigma_m(x)) &= \big(\lambda_2(0,m)\sigma_m(x)\big)(n,m\vee n)\sigma_{m\vee n}(\lambda_2(0,m)\sigma_m(x))\\&= \big(\lambda_2(0,m)\sigma_m(x)\big)(n,m\vee n)\sigma_{m\vee n}(x)\\&= \big(\mu_2(n,d(\mu_2))\rho\big)(n,m\vee n)\sigma_{m\vee n}(x),\end{aligned}$$

and then we have

$$\begin{aligned}\sigma_{m\vee n}(x) \in & D_{\mu_2\rho(m\vee n,d(\mu_2\rho))} \setminus D_{\mu_2\rho(m\vee n,d(\mu_2\rho))J_{\rho,\beta}} \\ & \implies \sigma_n(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_2\rho(n,d(\mu_2\rho))} \setminus D_{\mu_2\rho(n,d(\mu_2\rho))J_{\rho,\beta}} \\ & \implies \sigma_n(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_2(n,d(\mu_2))} \setminus D_{\mu_2(n,d(\mu_2))G_2}.\end{aligned}$$

So (28) is satisfied.

To prove the final part of the result, recall that, given $x \in \partial\Lambda^{\geq m\vee n}$ satisfying (27) and (28), we have the following formula for the pair (ρ,τ) in the set $\Lambda^{\min}(\mu_2(n,d(\mu_2)),\mu_1(n,d(\mu_1))\beta)$:

$$(\rho,\tau) = \Big(\sigma_n(\lambda_2(0,m)\sigma_m(x))^{\mu_1(n,d(\mu_1))\beta}_{\mu_2(n,d(\mu_2))}, \sigma_n(\lambda_2(0,m)\sigma_m(x))^{\mu_2(n,d(\mu_2))}_{\mu_1(n,d(\mu_1))\beta}\Big).$$

We then have

$$\begin{aligned}\mu_2(0,n)\sigma_n(\lambda_2(0,m)\sigma_m(x)) &= \mu_2(0,n)\big(\sigma_n(\lambda_2(0,m)\sigma_m(x))\big)(0,m\vee n-n)\sigma_{m\vee n}(x) \\ &= \mu_2(0,n)\big(\mu_2(n,d(\mu_2))\rho\big)(0,m\vee n-n)\sigma_{m\vee n}(x) \\ &= \mu_2\rho(0,m\vee n)\sigma_{m\vee n}(x). \qquad \square\end{aligned}$$

Proof of Proposition 6.1. We have already established Equation (26). Since Λ is finitely aligned, the sums in (26) are finite, and so

$$\iota_m^{m\vee n}\left(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)}\right)\iota_n^{m\vee n}\left(\Theta_{(\mu_1,G_1),(\mu_2,G_2)}\right) \in \mathcal{K}(X_{m\vee n}),$$

for every $m,n \in \mathbb{N}^k$, $(\lambda_1,F_1),(\lambda_2,F_2) \in \mathcal{A}^m$ and $(\mu_1,G_1),(\mu_2,G_2) \in \mathcal{A}^n$. It then follows from Proposition 4.5 that $\iota_m^{m\vee n}(\Theta_{x_1,x_2})\,\iota_n^{m\vee n}(\Theta_{y_1,y_2}) \in \mathcal{K}(X_{m\vee n})$, for every $x_1,x_2 \in X_m$ and $y_1,y_2 \in X_n$. Hence, $\iota_m^{m\vee n}(S)\,\iota_n^{m\vee n}(T) \in \mathcal{K}(X_{m\vee n})$, for every $S \in \mathcal{K}(X_m)$ and $(T \in \mathcal{K}(X_n)$. $\square$

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